Multipartite entanglement and its experimental detection

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Outline

1. **Motivation**
   - Why many-body entanglement is important?
2. **Different types of multipartite entanglement**
   - Two qubits
   - The three-qubit case in detail
   - Multipartite entanglement
3. **Systems with few particles**
   - Physical systems
   - Interesting quantum states
   - Witness design
   - Experiment
4. **Systems with very many particles**
   - Physical systems
   - Spin squeezing
   - Generalized spin squeezing
   - An experiment

Why is multipartite entanglement interesting?

- There have been many experiments recently aiming to create many-body entangled states.
- Quantum Information Science can help to find good targets for such experiments.
- Multipartite entangled states are needed in applications such as metrology.

Outline

Two qubits

Fact

Remember: There is only a single type of two-qubit entanglement.

- From many copies of entangled states, we can always distill a singlet using Local Operations and Classical Communication (LOCC).

- From any entangled two-qubit state, we can get to any other entangled two-qubit state through Stochastic Local Operations and Classical Communication (SLOCC).

  That is, for any entangled $|\Psi\rangle$ and $|\Phi\rangle$, there are invertible A and B such that
  $$|\Psi\rangle = A \otimes B|\Phi\rangle.$$

  Note that A and B do not have to be Hermitian.

Outline

Three-qubit pure states

- $|\Psi\rangle$ and $|\Phi\rangle$ are equivalent under SLOCC if there are invertible A, B and C such that
$$|\Psi\rangle = A \otimes B \otimes C|\Phi\rangle.$$

Four classes of states without genuine multipartite entanglement:

Class #1: three-qubit pure product state $|\Psi\rangle_1 \otimes |\Psi\rangle_2 \otimes |\Psi\rangle_3$

Class #2: biseparable states of the type $|\Psi\rangle_1 \otimes |\Psi\rangle_{2,3}$, and $|\Psi\rangle_{2,3}$ is entangled

Class #3: biseparable states of the type $|\Psi\rangle_{1,2} \otimes |\Psi\rangle_3$, and $|\Psi\rangle_{1,2}$ is entangled

Class #4: biseparable states of the type $|\Psi\rangle_{1,3} \otimes |\Psi\rangle_2$, and $|\Psi\rangle_{1,3}$ is entangled

Three-qubit pure states II

Two classes of genuine multipartite entanglement for pure states:

Class #5: W-class
The W state is defined as

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right).$$

Class #6: GHZ-class
The Greenberger-Horne-Zeilinger (GHZ) state is defined as

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right).$$

W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A 62, 062314 (2000)

Three-qubit pure states III

- A (GHZ $\cup$ W $\cup$ Bisep $\cup$ Sep)-class state, with an appropriate choice of basis states $|0\rangle$ and $|1\rangle$, has the general form

$$|\Psi_{GHZ}\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

  with $\lambda_k \geq 0$.

- A (W $\cup$ Bisep $\cup$ Sep)-class state has the form

$$|\Psi_W\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle.$$

- Thus, the (GHZ $\cup$ W $\cup$ Bisep $\cup$ Sep) class is the set of all physical states and the (W $\cup$ Bisep $\cup$ Sep)-class is a part of it.

Three-qubit pure states IV

Question:

Why are there two classes of genuine multipartite entanglement?

Answer:

The minimum number of product terms needed for a decomposition is

- 3 for W-class states and
- 2 for GHZ-class states.

This number cannot change by LOCC.

W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A 62, 062314 (2000)

Three-qubit mixed states

Six classes:

Class #1: fully separable states $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \varrho_3^{(k)}$

Class #2: (1)(23) biseparable states $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_{23}^{(k)}$, not in Class #1

Class #3: (12)(3) biseparable states $\sum_k p_k \varrho_{12}^{(k)} \otimes \varrho_3^{(k)}$, not in Class #1

Class #4: (13)(2) biseparable states $\sum_k p_k \varrho_{13}^{(k)} \otimes \varrho_2^{(k)}$, not in Class #1

Class #5: W-class states:
mxtr of pure (W $\cup$ Bisep $\cup$ Sep)-class states, not in Classes #1-4

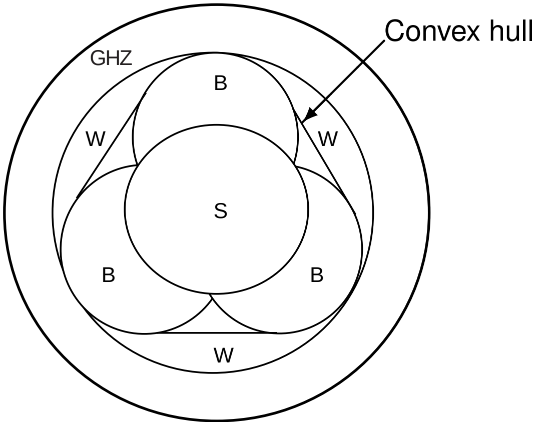
Class #6: GHZ-class states: mxtr of pure (GHZ $\cup$ W $\cup$ Bisep $\cup$ Sep)-class states, not in Classes #1-5

Biseparable states: mixture of states of classes #2, #3 and #4.

Three-qubit mixed states II

- The extension of the classification of pure states to mixed states leads to convex sets:

A. Acín, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001)



Witnesses for GHZ and W-class states

Entanglement witnesses for detecting states of a given class:

GHZ-class states

$$\mathcal{W}^{(P)}_{\mathrm{GHZ}} := \frac{3}{4}\mathbb{1} - |GHZ\rangle\langle GHZ|.$$

W-class states

$$\mathcal{W}^{(P)}_{\mathrm{W}} := \frac{2}{3}\mathbb{1} - |W\rangle\langle W|.$$

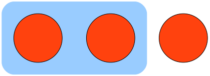
$$\mathcal{W}^{(P)}_{\mathrm{GHZ}} := \frac{1}{2}\mathbb{1} - |GHZ\rangle\langle GHZ|.$$

A. Acín, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001)

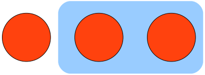
States that are biseparable with respect to all bipartitions

- There are states that are biseparable with respect to all the three bipartitions, but they are *not* fully separable.

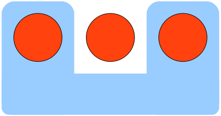
$$\varrho = \sum_k p_k \varrho_{12}^{(k)} \otimes \varrho_3^{(k)}$$



$$\varrho = \sum_k p'_k \varrho_{1}^{(k)} \otimes \varrho_{23}^{(k)}$$



$$\varrho = F_{12} \sum_k p''_k \varrho_{2}^{(k)} \otimes \varrho_{13}^{(k)} F_{12}$$



W. Dür, J.I. Cirac, Phys. Rev. A 61, 042314 (2000)

Outline

More than three qubits

- 4 qubits: There are 9 families and infinite number of SLOCC equivalence classes.
  [F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002)]
- For many qubits, the practically meaningful classification is
  - (Fully) separable
  - Biseparable entangled
  - Genuine multipartite entangled

More than three qubits II

Definition

A state is (fully) separable if it can be written as
$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}$.

Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

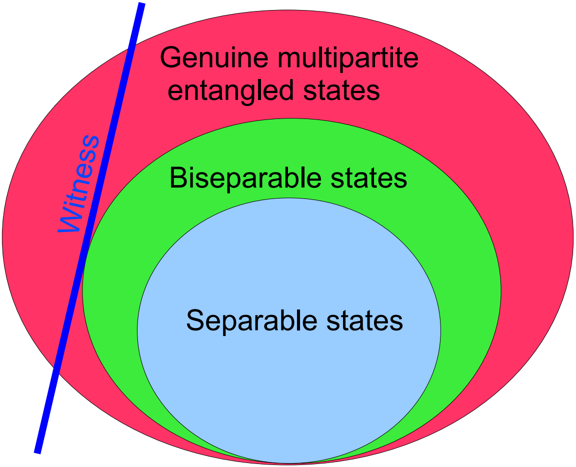
Here $|\Psi\rangle$ is an N-qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called genuine multi-partite entangled.

Convex sets for the multipartite case

- The idea of convex sets also work for the multi-qubit case: A state is biseparable if it can be obtained by mixing pure biseparable states.



Examples

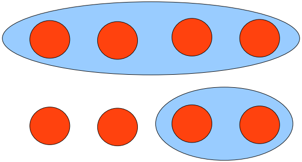
Examples

Two entangled states of four qubits:

$$|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

- The first state is genuine multipartite entangled, the second state is biseparable.

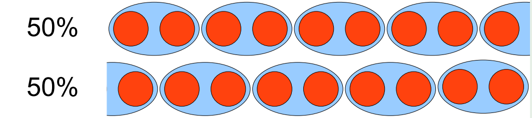


Other possible definition of genuine multipartite entanglement

- Alternative definition: a state is genuine multipartite entangled if it is inseparable with respect to all bipartitions.

Example

Mixture of the two biseparable states (chains of singlets)



It is inseparable with respect to all bipartitions.

- This state can be created in a two-qubit experiment.

Geometric measure of entanglement

- There is not a single entanglement measure for multipartite systems. ☛ **Talk by JENS SIEWERT**

Definition

For pure states, the geometric measure of entanglement is defined as

$$E_{\sin^2}(|\Psi\rangle) = 1 - \left(\max_{|\Psi_P\rangle \in \mathrm{PRODUCT}} \langle \Psi | \Psi_{\mathrm{P}} \rangle \right)^2 .$$

For mixed states, it is defined by a convex roof construction

$$E_{\sin^2}(\varrho) = \min_{\{|\Psi_k\rangle, p_k\}: \varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|} \sum_k p_k E_{\sin^2}(|\Psi_k\rangle).$$

- It is possible to calculate it for some pure states and for some mixed states.
T.-C. Wei, P.M. Goldbart, Phys. Rev. A 68, 042307 (2003)

Outline

Physical systems

State-of-the-art in experiments

- 8 qubits with trapped cold ions (Nature, 2005)
- 10 qubits with photons (Nature Physics, 2010)

Main Challenges

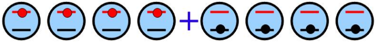
- How to obtain useful information when only *local* measurements are possible?
- *In principle, the entanglement witness method has the advantage that only one observable, the entanglement witness, needs to be measured. In practice, the measurement of this observable may be done by a series of local measurements. ... At this point the advantage over basic state tomography becomes somewhat questionable.*
  (B. TERHAL, IBM Watson Research Center, 2002)

Outline

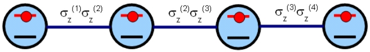
States

Quantum states in experiments:

- Greenberger-Horn-Zeilinger (GHZ) state or "Schrödinger cat state”



- Cluster state, graph state (obtained in Ising spin chains)



- Symmetric Dicke states

- Singlet states
  $(\Delta J_l)^2 = 0$ for $j = x, y, z.$

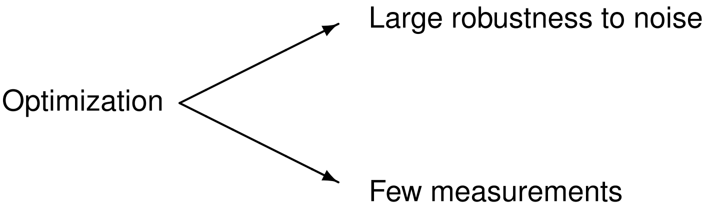
Aims when designing a witness

Definition

An entanglement witness $\mathcal{W}$ is an operator that is positive on all separable (biseparable) states.

Thus, $\mathrm{Tr}(\mathcal{W}\varrho) < 0$ signals entanglement (genuine multipartite entanglement). Horodecki 1996; Terhal 2000; Lewenstein, Kraus, , Cirac, Horodecki 2002

There are two main goals when searching for entanglement witnesses:



Robustness to noise

- A state mixed with white noise is given as

$$\varrho(p_{\text{noise}}) = (1 - p_{\text{noise}})\varrho + p_{\text{noise}}\varrho_{\text{noise}}$$

where p_{noise} is the ratio of noise and ϱ_{noise} is the noise. If we consider white noise then $\varrho_{\text{noise}} = \mathbb{1}/2^N$.

Definition

The noise tolerance of a witness $\mathcal{W}$ is characterized by the largest p_{noise} for which we still have

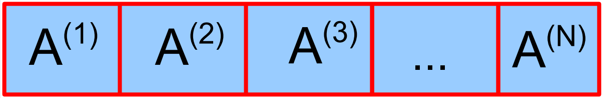
$$\text{Tr}(\mathcal{W}\varrho) < 0.$$

Only local measurements are possible

Definition

A single local measurement setting is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.



- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)}\rangle, \langle A^{(1)}A^{(3)}\rangle, \langle A^{(1)}A^{(2)}A^{(3)}\rangle ...$$

Decomposition of an operator

- All operators must be decomposed into the sum of locally measurable terms and these terms must be measured individually.
- For example,

$$
\begin{aligned}
|GHZ_3\rangle\langle GHZ_3| = & \frac{1}{8}(\mathbb{1} + \sigma_z^{(1)}\sigma_z^{(2)} + \sigma_z^{(1)}\sigma_z^{(3)} + \sigma_z^{(2)}\sigma_z^{(3)}) \\
& + \frac{1}{4}\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)} \\
& - \frac{1}{16}(\sigma_x^{(1)} + \sigma_y^{(1)})(\sigma_x^{(2)} + \sigma_y^{(2)})(\sigma_x^{(3)} + \sigma_y^{(3)}) \\
& - \frac{1}{16}(\sigma_x^{(1)} - \sigma_y^{(1)})(\sigma_x^{(2)} - \sigma_y^{(2)})(\sigma_x^{(3)} - \sigma_y^{(3)}).
\end{aligned}
$$

O. Gühne and P. Hyllus, Int. J. Theor. Phys. 42, 1001-1013 (2003). M. Bourennane et al., Phys. Rev. Lett. 92 087902 (2004)

- As N increases, the number of terms increases exponentially for projectors to quantum pure states.

Outline

Basic methods for designing witnesses

Three methods for designing witnesses:

- Projector witness, i.e., witness defined with the projector to a highly entangled quantum state
- Witness based on the projector witness
- Witness independent of the projector witness

Projector witness

- A witness detecting genuine multi-qubit entanglement in the vicinity of a pure state $|\Psi\rangle$ is

$$\mathcal{W}_{\Psi}^{(P)} := \lambda_{\Psi}^2 \mathbb{1} - |\Psi\rangle\langle\Psi|,$$

  where λ is the maximum of the Schmidt coefficients for $|\Psi\rangle$, when all bipartitions are considered.
  M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 2004

- A symmetric witness operator can always be decomposed as

$$P = \sum c_k A_k \otimes A_k \otimes A_k \otimes ... \otimes A_k.$$

- For symmetric operators, the number of settings needed is increasing polynomially with the number of qubits.
  GT, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, New J. Phys. 2009

Projector witness II

- GHZ states (robustness to noise is $\frac{1}{2}$ for large N!)

$$\mathcal{W}^{(P)}_{\text{GHZ}} := \frac{1}{2}\mathbb{1} - |GHZ_N\rangle\langle GHZ_N|.$$

- Cluster states

$$\mathcal{W}^{(P)}_{\text{CL}} := \frac{1}{2}\mathbb{1} - |CL_N\rangle\langle CL_N|.$$

- Dicke state

$$\mathcal{W}^{(P)}_{\text{D(N,N/2)}} := \frac{1}{2}\frac{N}{N-1}\mathbb{1} - |D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|.$$

- W-state

$$\mathcal{W}^{(P)}_{\text{W}} := \frac{N-1}{N}\mathbb{1} - |D_N^{(1)}\rangle\langle D_N^{(1)}|.$$

Witnesses based on the projector witness

- We construct witnesses that are easier to measure than the projector witness.

- Idea: If $\mathcal{W}^{(P)}$ is the projector witness and

$$\mathcal{W} - \alpha \mathcal{W}^{(P)} \geq 0$$

is fulfilled for some $\alpha > 0$, then $\mathcal{W}$ is also a witness.
GT and O. Gühne, Phys. Rev. Lett. and Phys. Rev. A 2005

Witnesses based on the projector witness II

Example

Witness requiring only two measurement settings for GHZ states

$$\mathcal{W}_{\mathrm{GHZ}}^{(P)} := \frac{1}{2}\mathbb{1} - |GHZ_N\rangle\langle GHZ_N|$$

$$\leq \mathcal{W}_{\mathrm{GHZ}}^{(P2)} := \mathbb{1} - \frac{1}{2} X_1 X_2 X_3 ... X_N - \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & ... & & \\ & & & 0 & \\ & & & & 1 \end{bmatrix}.$$

Measurement settings ⇒ [X X X X ...] [Z Z Z Z ...]

- Any state detected by $\mathcal{W}_{\mathrm{GHZ}}^{(P2)}$ is also detected by $\mathcal{W}_{\mathrm{GHZ}}^{(P)}$.
  GT and O. Gühne, Phys. Rev. Lett. and Phys. Rev. A 2005

Witnesses independent from the projector witness

- Witnesses without any relation to the projector witness.
- With an easily measurable operator M, we make a witness of the form

$$\mathcal{W} := c\mathbb{1} - M,$$

  where c is some constant.
- We have to set c to

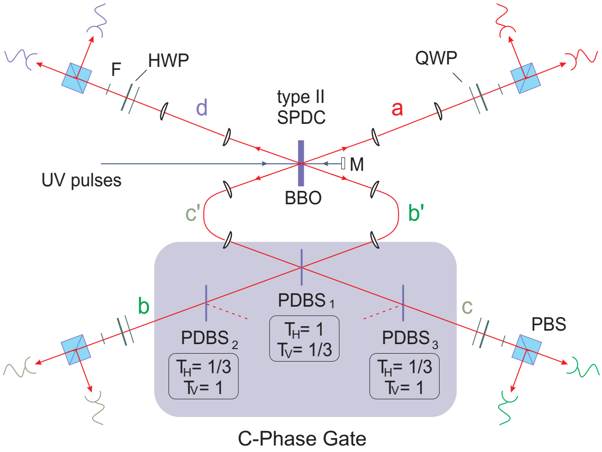
$$c = \max_{|\Psi\rangle \in \mathcal{B}} \langle M \rangle_{|\Psi\rangle},$$

  where $\mathcal{B}$ is the set of biseparable states. This problem is typically hard to solve.

Outline

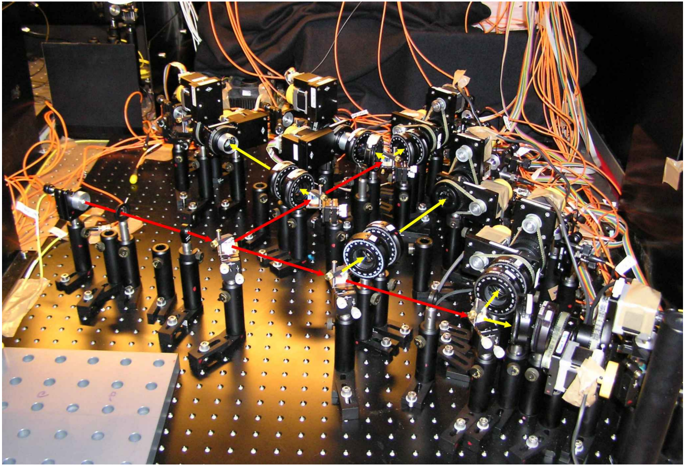
An experiment: Cluster state with photons

Experiment for creating a four-photon cluster state (Weinfurter group, 2005)



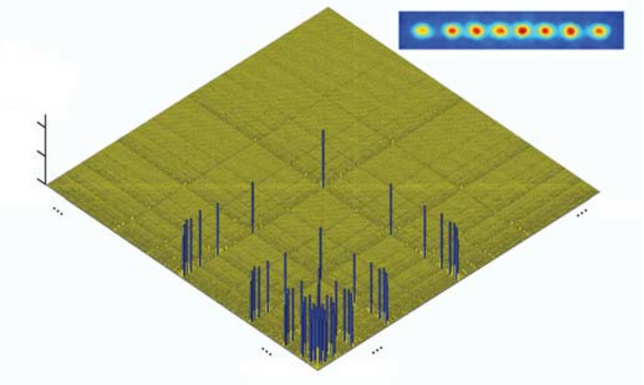
An experiment: Dicke state with photons

A photo of a real experiment (six-photon Dicke state, Weinfurter group, 2009):



Experiment: W-state with ions

- Experimental observation of an 8-qubit W-state with trapped ions.



H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, Nature 438, 643-646 (2005).

Quantum state tomography

- The density matrix can be recostructed from 3^N measurement settings.

- The measurements are

1. XXXX

2. XXXY

3. XXXZ

...

3^4. ZZZZ

- Note again that the number of measurements scales exponentially in N.

Outline

Physical systems

State-of-the-art in experiments

- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with 10^6 - 10^{12} atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.

Many-particle systems

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2\rangle - \langle J_l\rangle^2$$

variances.

Outline

Spin squeezing

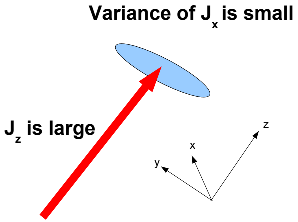
Definition

Uncertainty relation for the spin coordinates

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2.$$

If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_z\rangle|$ then the state is called spin squeezed (mean spin in the z direction!).
[M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993)]

☛ **Talk by ALICE SINATRA**



Spin squeezing II

Definition

Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

- Application: Quantum metrology, magnetometry.

☛ **Talk by AGOSTO SMERZI**

[A. Sørensen *et al.*, Nature **409**, 63 (2001)]

Outline

Complete set of the generalized spin squeezing criteria

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$

$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

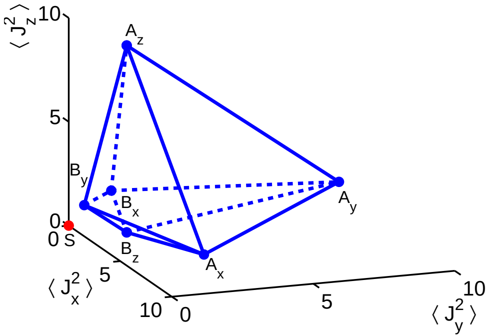
- Then any state violating the following inequalities is entangled

$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq N(N+2)/4, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq N/2, \\
\langle J_k^2 \rangle + \langle J_l^2 \rangle - N/2 &\leq (N-1)(\Delta J_m)^2, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + N(N-2)/4.
\end{aligned}$$

where k, l, m takes all the possible permutations of x, y, z.
[GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007]

The polytope

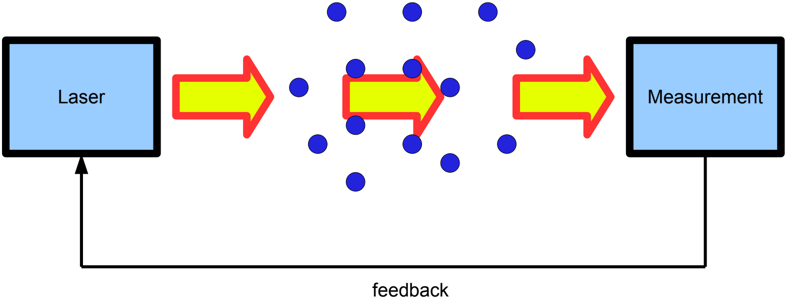
- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!



Outline

The physical system

- Rb atoms + light

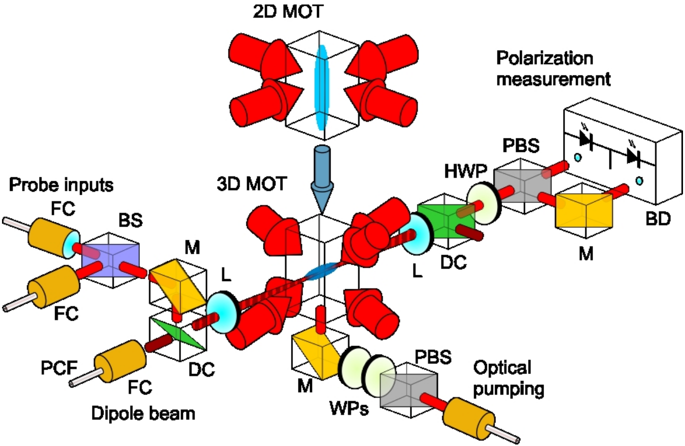


Experimental details

- Atoms interact with light. The light is measured, projecting the atoms into a squeezed state.
- Room temperature experiments: 10^{12} atoms
  [B Julsgaard, A Kozhekin, ES Polzik, Nature 2001].
  - Vapor cells
- Cold atom experiments: 10^6 atoms.
  - Laser cooling, sample in an optical dipole trap.
  - Atoms are transferred from a MOT to a dipole trap.

An experiment

Spin squeezing in a cold atomic ensemble (not BEC!)



Picture from M.W. Mitchell, ICFO, Barcelona.

Summary

- We discussed entanglement detection in multipartite systems.
- We considered
  - systems with few particles in which the particles could be individually addressed.
  - systems with very many particles, without the possibility of individual addressing

Review: O. Gühne and GT, “Entanglement detection”,

Physics Reports 474, 1-75 (2009).

THANK YOU FOR YOUR ATTENTION!