Entanglement detection with witness operators

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Outline

1. Motivation
2. Definition of entanglement
3. Witnesses
4. Multi-qubit witnesses with few measurements

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Motivation

- Due to the rapid development of quantum control today it is possible to create multi-qubit quantum states in several physical systems. Quantum information science must help to explore the large state space that can be accessed in such experiments.
- The notions "entanglement" and "genuine multi-qubit entanglement" help us to show the part of this state space that is "interesting" and can help to obtain fundamentally non-classical phenomena.

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Bell inequalities

- Bell inequalities are used to detect correlations that cannot come from a local hidden variable model. An example is the CHSH inequality: For local hidden variable models

$$\langle x_1 x_2 + x_1 y_2 - y_1 x_2 + y_1 y_2 \rangle \leq 2. \quad (1)$$

  Here we imagine measuring correlations on a bipartite system. At both parties, we measure two variables: x and y. At these measurements each time we get $+1$ or -1.
- There is a two-qubit quantum state that gives $\sqrt{2}$ for the left hand side of this inequality, if we interpret it as a quantum mechanical expectation value for the corresponding correlation operators.

Definition of Entanglement

- A bipartite state is called separable if it can be written as a convex combination of product states

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}, \tag{2}$$

  where $p_k > 0$ and $\sum_k p_k = 1$. Otherwise the state is called entangled.
- This definition comes from a 1989 paper of Werner. He also showed that there are entangled states for which all von Neumann measurements can be described by a local hidden variable model.
- Thus every nonlocal state is entangled, but not all entangled states are nonlocal.

Multipartite entanglement

- A multipartite state is called separable if it can be written as a convex combination of product states

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes ... \otimes \rho_k^{(N)}, \qquad (3)$$

  where $p_k > 0$ and $\sum_k p_k = 1$. Otherwise the state is called entangled.

- Is it a very useful definition? Not really. Take the state containing 100 particles

$$\rho_{100} := \rho_{\text{singlet}} \otimes |0\rangle\langle 0|^{\otimes 98}, \qquad (4)$$

  where $\Psi_{\text{singlet}} = (|01\rangle - |10\rangle)/\sqrt{2}$. Can one now claim that this is a 100 particle entangled state?

- Other example: many-qubit GHZ state stays entangled even when a lot of noise is added.

Possible approaches to multi-partite entanglement

- An N-qubit state is N-qubit entangled if it is not separable with respect to all its bipartitions (e.g., W. Dür). Problem: does not lead to a convex set of states.
- For example, let us define the state

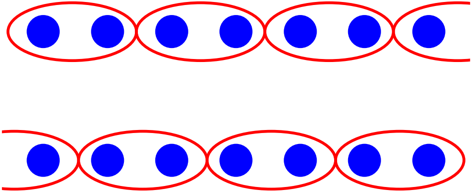
$$\rho_{\text{singlet chain}} = \rho_{\text{singlet}}^{\otimes \frac{N}{2}}. \tag{5}$$

This state is not N-qubit entangled according to the previous defintion. However,

$$\rho_{\text{mixed}} := \frac{1}{2}\left(\rho_{\text{singlet chain}} + S\rho_{\text{singlet chain}}S^{\dagger}\right) \tag{6}$$

is N-qubit entangled. (S is an operator shifting all the qubits to the right.)

Bicycle chain state (Wootters)



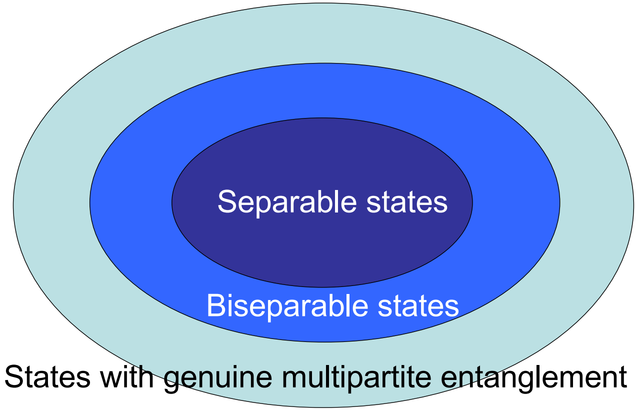
Genuine multipartite entanglement

- In a multi-qubit experiment it is important to detect genuine multi-qubit entanglement [Acín et. al, PRL **87**, 040401 (2001)]: We have to show that all the qubits were entangled with each other, not only some of them.
- An example of the latter case is a state of the form

$$|\Psi\rangle = |\Psi_{1..m}\rangle \otimes |\Psi_{m+1..N}\rangle. \tag{7}$$

  Here $|\Psi_{1..m}\rangle$ denotes the state of the first m qubits while $|\Psi_{m+1..N}\rangle$ describes the state of the remaining qubits. Such states are called biseparable.
- These concepts can be extended to mixed states. A mixed state is biseparable if it can be created by mixing biseparable pure states of the form Eq. (7).
- An N-qubit state is said to have genuine N-partite entanglement if it is not biseparable.

Convex sets


Separable states
Biseparable states
States with genuine multipartite entanglement

Outline

Entanglement detection

- We have convex sets. Thus we can use linear conditions to detect entanglement and genuine-mulitqubit entanglement.
- These linear conditions can be expressed using entanglement witness operators.
- W is an entanglement witness operator, if for every separable state $\rho_{\rm sep}$

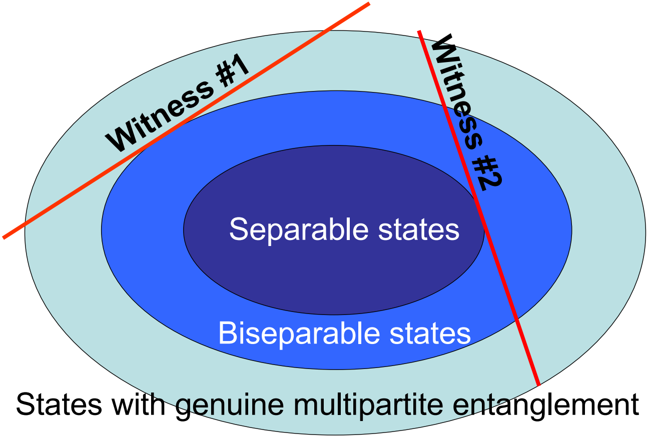
$$tr(\rho_{\rm sep} W) \geq 0, \tag{8}$$

  and for some entangled state $\rho_{\rm ent}$

$$tr(\rho_{\rm ent} W) < 0. \tag{9}$$

- Every entangled state can be detected by a witness. [Horodecki 1996]
- In practice, W is an observable. If its expectation value is negative, then we know that the state is entangled.

Witnesses and convex sets



Witness for detecting bipartite entanglement

- Let us construct a witness that detects entangled states in the vicinity of state $|\Psi\rangle$. Let us denote $|\phi_-\rangle$ the eigenvector corresponding to a negative eigenvalue λ_- of $|\Psi\rangle\langle\Psi|^{T1}$. Here $T1$ is transposition with respect to subsytem 1. Then the following operator is a witness [Lewenstein et al. 2000]

$$W_\Psi = |\phi_-\rangle\langle\phi_-|^{T1}. \tag{10}$$

- Proof. (1) It is non-negative on product states

$$tr(\rho_1 \otimes \rho_2 |\phi_-\rangle\langle\phi_-|^{T1}) = tr(\rho_1^T \otimes \rho_2 |\phi_-\rangle\langle\phi_-|) \geq 0 \tag{11}$$

  Due to convexity, this is also true for separable states.

- (2) It is negative on some entangled state. E.g.,

$$tr(|\Psi\rangle\langle\Psi| W_\Psi) = tr(|\Psi\rangle\langle\Psi| |\phi_-\rangle\langle\phi_-|^{T1}) = tr(|\Psi\rangle\langle\Psi|^{T1} |\phi_-\rangle\langle\phi_-|) = \lambda_- < 0$$

Optimal witnesses

- Which witness is better than the other?
- One possibility: a witness W_1 is finer than witness W_2 if for every ρ such that $tr(W_2\rho) < 0$ we have $tr(W_1\rho) < 0$.
[Lewenstein, Kraus, Cirac, and Horodecki PRA 2000]
- In other words, for some $\alpha > 0$

$$\alpha W_2 = W_1 + P, \tag{12}$$

where P is a positive semi-definite operator.
- W is optimal if there is not another witness that is finer than W.

Decomposable witnesses

- A witness is called decomposable if it can be written as [Lewenstein, Kraus, Cirac, and Horodecki, PRA 2000.]

$$W = P + Q^{T1}, \tag{13}$$

where $P, Q \geq 0$.

- Decomposable witnesses detect only entangled states with a non-positive partial transpose. *Proof.* Let ρ be a state with a positive semidefinite partial transpose. Then

$$tr(W\rho) = tr(P\rho) + tr(Q\rho^{T1}) \geq 0. \tag{14}$$

- The optimal decomposable witnesses have the form

$$W = Q^{T1}, \tag{15}$$

where $Q \geq 0$.

Multi-qubit witnesses

- Multi-qubit witnesses for detecting genuine multi-qubit entanglement are typically of the form

$$W = const. \times \mathbb{1} - |\Psi\rangle\langle\Psi|, \tag{16}$$

  where $|\Psi\rangle$ is a highly entangled state, e.g., a GHZ state. const. is the square of the maximum of the Schmidt coefficients of Ψ when all bipartitions are considered [Bourennane et al., PRL 2004].
- Examples:

$$\begin{aligned} W_{\mathrm{GHZN}} &= \frac{\mathbb{1}}{2} - |GHZ_N\rangle\langle GHZ_N|, \\ W_{\mathrm{CN}} &= \frac{\mathbb{1}}{2} - |C_N\rangle\langle C_N|, \\ W_{\mathrm{WN}} &= \frac{N-1}{N}\mathbb{1} - |W_N\rangle\langle W_N|. \end{aligned} \tag{17}$$

  Here $|GHZ_N\rangle$ is the GHZ state, $|C_N\rangle$ is the cluster state, and $|W_N\rangle$ is the W state.

Robustness to noise

- Idea behind these witnesses: In an experiment we aim to prepare the GHZ state. The preparation is never perfect, but the prepared state must be close to the GHZ state. Thus the previous fidelity based witnesses can detect the state as genuine multi-qubit entangled.
- It is important to know how large neighborhood of the state to be prepared is detected by our witness as entangled. This can be done by the robustness to white noise. Let us consider the state

$$\rho_{\mathrm{n}}(p_n) := p_n |GHZ_N\rangle\langle GHZ_N| + p_n \frac{\mathbb{1}}{2^N}. \tag{18}$$

  The robustness to noise is characterized by the largest p_n for which $\rho_{\mathrm{n}}(p_n)$ is still detected as entangled.
- For the GHZ state and cluster state witnesses for large N we have 0.5, for witnesses for all other states it is smaller.
- Note: Robustness to noise characterizes the witness not the state.

How to measure a multi-qubit witness

- Witnesses are very often of the form

$$W = const. \times \mathbb{1} - |\Psi\rangle\langle\Psi|, \tag{19}$$

where $|\Psi\rangle$ is a highly entangled state, e.g., a GHZ state. How can we measure such a projector?

- A solution is decomposing the witness into the sum of local terms

$$W = \sum_k A_k^{(1)} \otimes A_k^{(2)} \otimes A_k^{(3)} \otimes ... \otimes A_k^{(N)}. \tag{20}$$

- The number of terms needed increases very fast (probably exponentially) for many witnesses with the number of qubits. At this point of the talk, it is not clear that a ten qubit witness can ever be measured. [Gühne et al., Phys. Rev. A 66, 062305 (2002)]

Measurement setting

- Measurement setting: We measure simultaneously single-qubit operators, i.e., at each qubit k we measure $O^{(k)}$. After repeating this several times, any two-point, three-point, etc. correlations of the form $\langle O^{(k)}O^{(l)}\rangle$, $\langle O^{(k)}O^{(l)}O^{(m)}\rangle$ can be obtained.
- Thus the number of measurement settings determines the measurement effort, not the number of correlation terms in the decomposition of the witness.
- Example: How to measure $|GHZ_N\rangle\langle GHZ_N|$. It can be decomposed into

$$|GHZ_4\rangle\langle GHZ_4| = \frac{1}{2^N}[\mathbb{1} + Z^{(1)}Z^{(2)} + Z^{(1)}Z^{(2)}Z^{(3)}Z^{(4)} \\ +X^{(1)}X^{(2)}X^{(3)}X^{(4)} - Y^{(1)}Y^{(2)}X^{(3)}X^{(4)} + Y^{(1)}Y^{(2)}Y^{(3)}Y^{(4)}],$$

  where each term represents the sum of all its possible permutations. The first two terms can be measured with a single setting. Note that a better decomposition is also possible [Gühne et al., 2003.].

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Measuring a multi-qubit GHZ or cluster state witness

- Now I show a method to measure the projector-based witnesses presented before with only two measurement settings.
- Idea: For measuring (an estimation for) witness W, we look for an operator $\widetilde{W}$ such that

$$\widetilde{W} \geq W \tag{21}$$

  and $\widetilde{W}$ is easy to decompose.
- The idea we will present works for witnesses based on a projector to GHZ states, cluster states, and in general, graph states.

GHZ states, cluster states and graph states

- All these quantum states have one thing in common: For an N-qubit state there are 2^N local operators that have the state as an eigenvector with an eigenvalue $+1$. This group, called stabilizer, is commutative and the square of all group elements is $\mathbb{1}$ [Gottesman, PRA 1997].
- The projector to these states is can be written as

$$|\Psi\rangle\langle\Psi| = \frac{1}{2^N}\sum_{k=1}^{2^N} S_k = \frac{1}{2^N}\prod_{k=1}^{N}(\mathbb{1} + g_k). \quad (22)$$

  Here S_k are the group elements and g_k are the generators.
- At this point we can see that an N-qubit projector of these states can always be decomposed into the sum of 2^N local terms.

Concrete examples

- Generators for the stabilizer of the GHZ state:

$$\begin{aligned} g_1 &= X^{(1)}X^{(2)}X^{(3)}...X^{(N)}, \\ g_k &= Z^{(k-1)}Z^{(k)}, \end{aligned} \tag{23}$$

  for $2 \leq k \leq N$.
- Generators for the stabilizer of the cluster state [Raussendorf and Briegel, PRL 2001]:

$$\begin{aligned} g_1 &= X^{(1)}Z^{(2)}, \\ g_k &= Z^{(k-1)}X^{(k)}Z^{(k+1)}, \\ g_N &= Z^{(N-1)}X^{(N)}, \end{aligned} \tag{24}$$

  for $2 \leq k \leq N-1$.
- Generators of a graph state are [M. Hein, J. Eisert, and H.J. Briegel, PRA 2004]

$$g_k = X^{(k)} \prod_{l \in Neigh(k)} Z^{(k)}. \tag{25}$$

Stabilizer witnesses

- So we want to measure this witness:

$$W_{\mathrm{GHZN}} = \frac{\mathbb{1}}{2} - |GHZ_N\rangle\langle GHZ_N|. \qquad (26)$$

  We look for another witness $\widetilde{W}_{\mathrm{GHZN}}$ for which $\widetilde{W}_{\mathrm{GHZN}} \geq W_{\mathrm{GHZN}}$ and it is easy to measure.
- Idea: we look for this witness as a linear combination of stabilizer elements

$$\widetilde{W}_{\mathrm{GHZN}} = \sum_k c_k S_k. \qquad (27)$$

- We optimize $\{c_k\}$ such that $\widetilde{W}_{\mathrm{GHZN}}$ has the best possible robustness to noise.
- [Theory: Tóth and Gühne, PRL 2005; PRA 2005. Experiment: Kiesel et al., PRL 2005.]

Results for GHZ and cluster states

- For GHZ states the optimal witness looks like

$$\widetilde{W}_{\text{GHZN}} := \frac{3}{2}\mathbb{1} - \frac{1}{2}(1 + X^{(1)}X^{(2)}X^{(3)}...X^{(N)}) - \frac{1}{2^{N-1}}\prod_{k=2}^{N}(1 + Z^{(k-1)}Z^{(k)}). \tag{28}$$

- For cluster states the optimal witness looks like

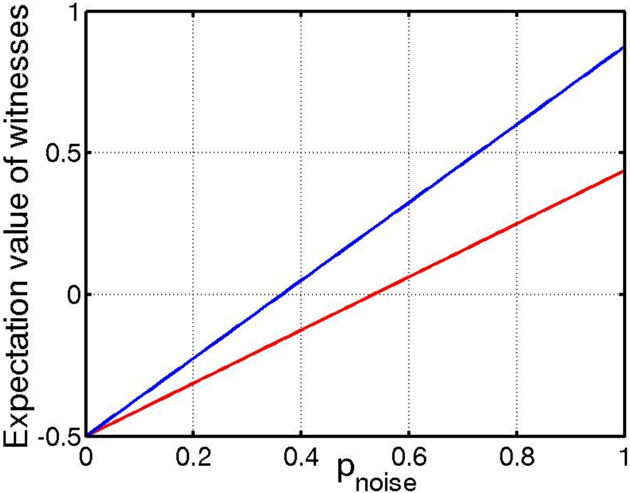
$$\begin{aligned}\widetilde{W}_{\text{CN}} &:= \frac{3}{2}\mathbb{1} - \frac{1}{2^{N/2}}\prod_{k=1,3,5,..}(1 + Z^{(k-1)}X^{(k)}Z^{(k+1)}) \\ &- \frac{1}{2^{N/2}}\prod_{k=2,4,6,..}(1 + Z^{(k-1)}X^{(k)}Z^{(k+1)}),\end{aligned} \tag{29}$$

where $Z^{(0)} = Z^{(N+1)} = \mathbb{1}$.

- Robustness to noise for large N : 33% for the GHZ witness and 25% for the cluster state witness.

Example for witnesses

- Let us consider a noisy four-qubit GHZ state. We plot the expectation values of the two witnesses as functions of the noise ratio p_{noise} :



Fidelity estimation

- The expectation values of our witnesses give information on the fidelity of our state with respect to GHZ and cluster states.
- For GHZ states the optimal witness looks like

$$|GHZ_N\rangle\langle GHZ_N| \geq \frac{\mathbb{1}}{2} - \widetilde{W}_{\mathrm{GHZN}}. \qquad (30)$$

- For cluster states the optimal witness looks like

$$|C_N\rangle\langle C_N| \geq \frac{\mathbb{1}}{2} - \widetilde{W}_{\mathrm{CN}}. \qquad (31)$$

Conclusion

- I gave an overview on using entanglement witnesses for detecting entanglement.
- I considered the problem of measuring a multi-qubit witness with local measurements.
- I presented some witnesses that need few measurements for detecting entanglement close to GHZ and cluster states.
- I showed that similar ideas work also for fidelity estimation.

THANK YOU FOR YOUR ATTENTION!!!

For more details please see http://optics.szfki.kfki.hu/~toth/.