Detection of multipartite entanglement in the vicinity of symmetric Dicke states (quant-ph/0511237)

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QIP Workshop, Pécs, 2006 July

Outline

1. Introduction
   - Motivation
   - Definition of Dicke states
   - Why are they useful for quantum information processing?
   - Genuine multipartite entanglement
2. Fidelity-based entanglement criterion
3. Entanglement detection with collective observables
4. Connection to super-radiance

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Motivation

- One of the main aims of quantum control is to look for and realize "interesting" highly entangled quantum states (like GHZ states and cluster states).
- "interesting" means that the state should have some properties which are useful from the point of view of quantum information processing. If we talk about experiments, it should also be easy to prepare it and detect entanglement around it.
- Symmetric Dicke states are highly entangled quantum states which appear naturally in quantum systems with many particles and a symmetric dynamics. We will argue that they also have some nice properties from the point if view of quantum information.
- Actuality: A four-photon Dicke state was produced in an experiment at the Max Planck Institute for Quantum Optics: [N. Kiesel, C. Schmid, G. Toth, E. Solano, and H. Weinfurter, quant-ph/0606234.] $\rightarrow$ Remember the talk of Nikolai Kiesel on this workhop.

Definition of Dicke states

- Dicke states are the simultaneous eigenstates of the collective angular momentum, J and its z-component, J_z.
- In a typical many-qubit experiment, in which the qubits cannot be individually accessed, both the initial state and the dynamics are symmetric under the permutation of qubits. Thus we will consider such symmetric Dicke states. These are also the states with maximal J. An N-qubit symmetric Dicke state with m excitations is defined as

$$|m, N\rangle := \binom{N}{m}^{-\frac{1}{2}} \sum_k P_k(|1_1, 1_2, ..., 1_m, 0_{m+1}, ..., 0_N\rangle), \tag{1}$$

  where $\{P_k\}$ is the set of all distinct permutations of the spins.
- In particular, we will consider the state $|N/2, N\rangle$ and show some nice properties of this state.

Dicke's original idea

- Consider N two-state atoms interacting with an electromagnetic field.
- Let us set all the atoms in the excited state

$$\Psi_0 := |11111...\rangle \equiv |N, N\rangle. \tag{2}$$

- If the atomic cloud emits coherently, then, after emitting a photon, it can go to the state

$$\Psi_1 := (|01111...\rangle + |10111...\rangle + |11011...\rangle + ...)/\sqrt{N} \equiv |N-1, N\rangle. \tag{3}$$

- After emitting further photons the system goes through the states $|N-2, N\rangle$,$|N-3, N\rangle$, ..., $|N/2, N\rangle$, ..., $|0, N\rangle$.
- Dicke found that the intensity of the radiation is proportional to N for the state $|11111...\rangle$, however, it is proportional to roughly N^2 for the state $|N/2, N\rangle$. This he called superradiance.
- Remember, $|N/2, N\rangle$ is the equal superposition of states with $N/2$ zeros and $N/2$ ones.

Why are Dicke states useful for quantum information processing?

- The state $|N/2, N\rangle$ has the maximal singlet fraction possible among N-qubit quantum states. Thus it is useful for telecloning. [See experimental paper at quant-ph/0606234.]
- Unlike GHZ states, the entanglement of the state $|N/2, N\rangle$ is robust against particle loss.
- We argue that this state is useful for the experimental creation and detection of multi-qubit entanglement.

Genuine multipartite entanglement

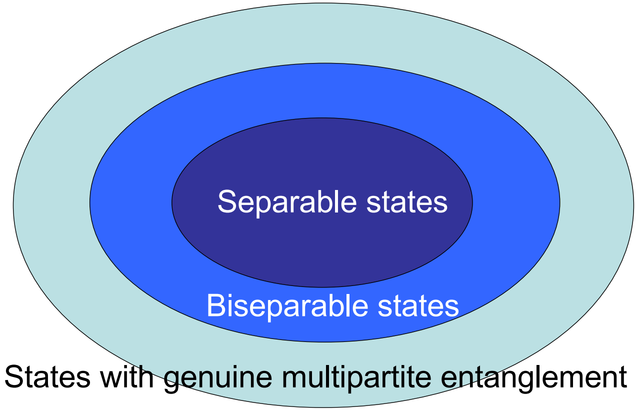
- In a multi-qubit experiment it is important to detect genuine multi-qubit entanglement [Acín et. al, PRL **87**, 040401 (2001)]: We have to show that all the qubits were entangled with each other, not only some of them.
- An example of the latter case is a state of the form

$$|\Psi\rangle = |\Psi_{1..m}\rangle \otimes |\Psi_{m+1..N}\rangle. \tag{4}$$

Here $|\Psi_{1..m}\rangle$ denotes the state of the first m qubits while $|\Psi_{m+1..N}\rangle$ describes the state of the remaining qubits. Such states are called biseparable.

- These concepts can be extended to mixed states. A mixed state is biseparable if it can be created by mixing biseparable pure states of the form Eq. (4).
- An N-qubit state is said to have genuine N-partite entanglement if it is not biseparable.

Convex sets


Separable states
Biseparable states
States with genuine multipartite entanglement

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Fidelity-based entanglement criteria

- Based on [M. Bourennane et. al, PRL **92**, 107901 (2004)] we know that for biseparable states ρ

$$Tr(\rho|\Psi\rangle\langle\Psi|) \leq C_\Psi. \quad (5)$$

  Here $|\Psi\rangle$ is a multi-qubit entangled state and C_Ψ is the square of the maximal overlap of $|\Psi\rangle$ with biseparable states
- This criterion can be used to detect entanglement in the vicinity of state $|\Psi\rangle$. Any quantum state ρ which violates this inequality is genuine multi-qubit entangled.
- Fortunately, it turns out that C_Ψ equals the square of the maximum of the Schmidt coefficients of $|\Psi\rangle$ with respect to any bipartition. Thus C_Ψ can be determined easily, without the need for multi-variable optimization.

The required fidelity for some states

The required fidelity for detecting genuine multi-qubit entanglement is the following for some well-known quantum states

- $C_{GHZ} = 1/2$,
- $C_{cluster} = 1/2$ [G. Tóth and O. Gühne, PRL 2005],
- $C_W = (N-1)/N$ [H. Häffner et. al, Nature **438**, 643 (2005)].

Clearly, it is hard to detect a W state this way, since the fidelity is approaching $+1$ as N increases. For the GHZ and cluster states, however, C does not increase with N and it is the smallest possible: $1/2$.

How much is C for our state $|N, N/2\rangle$?

Theorem 1. For biseparable quantum states

$$Tr(\rho|N/2, N\rangle\langle N/2, N|) \leq \frac{1}{2}\frac{N}{N-1} =: C_{N/2,N}. \tag{6}$$

Thus for large N the required fidelity is around $1/2$.

Proof of Theorem 1.

The Schmidt decomposition of $|m, N\rangle$ according to the partition $(1, 2, ..., N_1)(N_1 + 1, N_1 + 2, ..., N)$ is [J.K. Stockton et al., PRA 2003].

$$|m, N\rangle = \sum_k \lambda_k |k, N_1\rangle \otimes |m - k, N - N_1\rangle \tag{7}$$

where the Schmidt coefficients are

$$\lambda_k = \begin{pmatrix} N \\ m \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} N_1 \\ k \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} N - N_1 \\ m - k \end{pmatrix}^{\frac{1}{2}}. \tag{8}$$

We do not have to consider other partitions due to the permutational symmetry of our Dicke states. For an N-qubit symmetric Dicke state with $N/2$ excitations we have $m = N/2$. Now we will need to know that

$$\begin{pmatrix} N_1 \\ k \end{pmatrix} \begin{pmatrix} N - N_1 \\ \frac{N}{2} - k \end{pmatrix} \leq \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} N - 2 \\ \frac{N}{2} - 1 \end{pmatrix}. \tag{9}$$

The proof of Eq. (9) can be found in quant-ph/0511237. Thus we find that the maximal Schmidt coefficient can be obtained for $N_1 = 2$ and $k = 1$, and it is $N(N - 1)/2$. □

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Entanglement detection with collective observables

- In certain physical systems (e.g., optical lattices of bosonic two-state atoms) only the measurement of collective quantities is possible.
- Thus entanglement criteria are needed which detect entanglement with collective measurements, without the need of accessing the qubits individually.
- These criteria are typically built with the collective spin operators

$$J_{x/y/z} := \frac{1}{2} \sum_{k=1}^{N} \sigma_{x/y/z}^{(k)}, \tag{10}$$

where $\sigma_{x/y/z}^{(k)}$ denote Pauli spin matrices acting on qubit k.

Entanglement detection with collective observables - Previous Work

- *Spin squeezing criterion* [A. Sørensen et al. , Nature **409**, 63 (2001)]. For separable states

$$\frac{(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} \geq \frac{1}{N}. \qquad (11)$$

  Any state violating this condition is entangled.

- *Entanglement detection around a singlet* [G. Tóth, Phys. Rev. A **69**, 052327 (2004)]. For separable states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq 2N. \qquad (12)$$

  The left hand side is minimal for many-body singlets.

Lemma

Lemma 1. For separable states the maximum of the expression

$$a_x\langle J_x^2\rangle + a_y\langle J_y^2\rangle + a_z\langle J_z^2\rangle + b_x\langle J_x\rangle + b_y\langle J_y\rangle + b_z\langle J_z\rangle \qquad (13)$$

with $a_{x/y/z} \geq 0$ and real $b_{x/y/z}$ is the same as its maximum for translationally invariant product states (i.e., for product states of the form $|\Psi\rangle = |\psi\rangle^{\otimes N}$).

Proof 1. Simple multivariable minimisation using a clever constraint.

- Importance of Lemma 1: Usually it is very hard to find the maximum/minimum of the expectation value of an operator for separable states.
- See [J. Eisert, P. Hyllus, O. Gühne, and M. Curty, PRA **70**, 062317 (2004); A.C. Doherty, P.A. Parrilo, and F.M. Spedalieri, PRA **71**, 032333 (2005).]

Simple criterion

Theorem 2. As a special case of the previous criterion, we have that for separable states

$$\langle J_x^2\rangle + \langle J_y^2\rangle \leq \frac{N}{2}\left(\frac{N}{2}+\frac{1}{2}\right). \tag{14}$$

For even N, the left hand side is the maximal $\frac{N}{2}\left(\frac{N}{2}+1\right)$ only for an N-qubit symmetric Dicke state with $N/2$ excitations.
Proof. Based on Lemma 1, the proof of this theorem is obvious.

- It can be seen that the bound in Eq. (14) is sharp since a separable state of the form

$$|\Psi_{\mathrm{xy}}\rangle := (|0\rangle + |1\rangle e^{i\phi})^{\otimes N} \tag{15}$$

for any real ϕ saturates the bound.

Entanglement detection based on the maximum of variances for separable states

- Based on Eq. (14), it is easy to see that for separable states we also have

$$(\Delta J_x)^2 + (\Delta J_y)^2 \leq \frac{N}{2}\left(\frac{N}{2} + \frac{1}{2}\right). \tag{16}$$

  Thus $J_{x/y}^2$ could be replaced by the corresponding variances.
- Any state violating Eq. (16) is entangled. Note the curious nature of our criterion: A state is detected as entangled, if the uncertainties of the collective spin operators are larger than a bound.

Detecting genuine multipartite entanglement

It is also possible to detect genuine multipartite entanglement with similar methods. Such a criterion has already been presented for three qubits in [G. Tóth and O. Gühne, PRA **72**, 022340 (2005)]. For biseparable three-qubit states

$$3/2 + Q/2 := \langle J_x^2 \rangle + \langle J_y^2 \rangle \leq 2 + \sqrt{5}/2 \approx 3.12. \qquad (17)$$

Here X and Y are Pauli spin matrices.
Proof. Let us assume (1)(23) biseparability Then for a state of the form $\Psi = \Psi_1 \otimes \Psi_{23}$

$$\begin{aligned}
\langle Q \rangle &= \langle X^{(1)} \rangle \langle X^{(2)} \rangle + \langle X^{(1)} \rangle \langle X^{(3)} \rangle + \langle X^{(2)} X^{(3)} \rangle \\
&+ \langle Y^{(1)} \rangle \langle Y^{(2)} \rangle + \langle Y^{(1)} \rangle \langle Y^{(3)} \rangle + \langle Y^{(2)} Y^{(3)} \rangle \\
&= \langle X^{(1)} \rangle_1 [\langle X^{(2)} + X^{(3)} \rangle_{23}] + \langle Y^{(1)} \rangle_1 [\langle Y^{(2)} + Y^{(3)} \rangle_{23}] \\
&- [\langle X^{(2)} X^{(3)} + Y^{(2)} Y^{(3)} \rangle_{23}] \\
&= \langle F(x,y) \rangle_{23}. \qquad (18)
\end{aligned}$$

Here

$$x := \langle X^{(1)} \rangle_1; \quad y := \langle Y^{(1)} \rangle_1. \tag{19}$$

$\langle ... \rangle_1$ and $\langle ... \rangle_{23}$, respectively, denote expectation value computed for Ψ_1 and Ψ_{23}. To be explicit, matrix F, as the function of two real parameters, is given

$$\begin{aligned} F(x,y) &:= x[X^{(2)} + X^{(3)}] + y[Y^{(2)} + Y^{(3)}] \\ &+ [X^{(2)}X^{(3)} + Y^{(2)}Y^{(3)}]. \end{aligned} \tag{20}$$

The expectation value of $F(x,y)$ can be bounded from below

$$\langle F(x,y) \rangle_{23} \le \Lambda_{\max}[F(x,y)], = \tag{21}$$

where $\Lambda_{\max}(F)$ is the largest eigenvalue of F. Using $\langle X^{(1)} \rangle^2 + \langle Y^{(1)} \rangle^2 \le 1$, we obtain that $\langle Q \rangle$ is bounded from above by

$$\langle Q \rangle \le \max_{x^2+y^2 \le 1} \{\Lambda_{\max}[F(x,y)]\} = 1 + \sqrt{5}. \tag{22}$$

This is clearly valid also for general biseparable states. □

Four-qubit case

Theorem 3. For a four-qubit biseparable state

$$\left\langle J_x^2 \right\rangle + \left\langle J_y^2 \right\rangle \leq 7/2 + \sqrt{3} \approx 5.23. \tag{23}$$

For the left hand side of Eq. (23) the maximum is 24 and it is obtained uniquely for the $|2, 4\rangle$ state.
Proof. Similar to the three-qubit case.

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Connection to superradiance

- The operator to be measured for our criterion is identical to the one which appeared in Dicke's original paper in 1954 as the intensity of the superradiant light during spontaneous emission in a cloud of atoms.
- To be more precise, the light intensity is

$$I := I_0 \left\langle J_x^2 + J_y^2 + J_z \right\rangle, \tag{24}$$

  where I_0 is the radiation rate of one atom in its excited state. Our entanglement criterion shows that if $I/I_0 - \langle J_z \rangle$ is larger than a bound then the system is entangled.
- Unexpectedly, there are separable states [e.g., the state $|1111...\rangle_x$] for which the light intensity scales roughly with the square of the number of qubits. We have also shown that there was similar bound for the intensity for multipartite entanglement.

Conclusion

1. We discussed the advantages of using highly entangled symmetric Dicke states for quantum information processing.
2. We showed that they are very well suited for the experimental creation and detection of multipartite entanglement. We discussed
   - fidelity-based entanglement criteria, and
   - criteria based on measuring collective observables.
3. We discussed the connection of our entanglement detection scheme to super-radiance.
4. For further details please see quant-ph/0511237.

******** THANK YOU!!! *********