

**Optimal spin squeezing inequalities detect  
bound entanglement in spin models  
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# Motivation

- In many quantum control experiments the qubits cannot be individually accessed. We still would like to detect entanglement.
- The spin squeezing criterion is already known. Are there other similar criteria that detect entanglement with the first and second moments of collective observables?
- Generalized spin squeezing inequalities might help identifying the entangled states useful for metrology.

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# Spin squeezing

- 1 Spin squeezing, according to the original definition, is interpreted in the following context. The variances of the angular momentum components are bounded by the following uncertainty relation

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z \rangle|^2.$$

If  $(\Delta J_x)^2$  is smaller than the standard quantum limit  $\frac{|\langle J_z \rangle|}{2}$  then the state is called spin squeezed.

- 2 In practice this means that the angular momentum of the state has a small variance in one direction, while in an orthogonal direction the angular momentum is large.

[M. Kitagawa and M. Ueda, PRA **47**, 5138 (1993).]

# Definition of entanglement

- **Fully separable** states are states that can be written in the form

$$\rho = \sum_I p_I \rho_I^{(1)} \otimes \rho_I^{(2)} \otimes \dots \otimes \rho_I^{(N)},$$

where  $\sum_I p_I = 1$  and  $p_I > 0$ .

- A state is **entangled** if it is not separable.
- Note that one could also look for other type of entanglement in many-particle systems, e.g., entanglement in the two-qubit reduced density matrix.

# Collective quantities

- What if we cannot address the particles individually? This is expected to occur often in future experiments.
- For spin- $\frac{1}{2}$  particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)},$$

where  $l = x, y, z$  and  $\sigma_l^{(k)}$  a Pauli spin matrices. We can also measure the  $(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$  variances.



# The standard spin-squeezing criterion

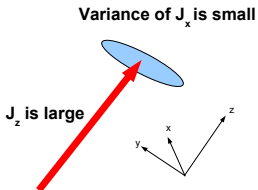
- The spin squeezing criteria for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature **409**, 63 (2001).]

- States violating it are like this:



# Generalized spin squeezing entanglement criteria I

- Separable states must fulfill

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}.$$

It is maximally violated by a many-body singlet, e.g., the ground state of an anti-ferromagnetic Heisenberg chain.

[GT, PRA **69**, 052327 (2004).]

- For such a state

$$\langle J_k^m \rangle = 0.$$

- Note that there are very many states giving zero for the left hand side. The mixture of all such states also maximally violates the criterion.

# Generalized spin squeezing entanglement criteria II

- For states with a separable two-qubit density matrix

$$\left( \langle J_k^2 \rangle + \langle J_l^2 \rangle - \frac{N}{2} \right)^2 + (N-1)^2 \langle J_m \rangle^2 \leq \langle J_m^2 \rangle + \frac{N(N-2)}{4}$$

holds.

[J. Korbicz, I. Cirac, M. Lewenstein, PRL **95**, 120502 (2005).]

- Detects all symmetric two-qubit entangled states; can be used to detect symmetric Dicke states.
- Used in ion trap experiment.

[J. Korbicz, O. Gühne, M. Lewenstein, H. Häffner, C.F. Roos, R. Blatt, PRA **74**, 052319 (2005).]

# Generalized spin squeezing entanglement criteria III

- For separable states [GT, J. Opt. Soc. Am. B **24**, 275 (2007).]

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle \leq \frac{N(N+1)}{4}$$

holds.

- This can be used to detect entanglement close to  $N$ -qubit symmetric Dicke states with  $\frac{N}{2}$  excitations. For such a state

$$\begin{aligned}\langle J_k \rangle &= 0, \\ \langle J_z^2 \rangle &= 0, \\ \langle J_{x/y}^2 \rangle &= \frac{N(N+2)}{8}.\end{aligned}$$

- For  $N = 4$ , this state looks like

$$|\Psi\rangle = \frac{1}{\sqrt{6}}(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle).$$

This was realized with photons.

[N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL **98**, 063604 (2007).]

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# Optimal spin squeezing inequalities

- Let us assume that for a system we know only

$$\mathbf{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$

$$\mathbf{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled

$$\begin{aligned}\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq \frac{N(N+2)}{4}, \\ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \frac{N}{2}, \\ \langle J_k^2 \rangle + \langle J_l^2 \rangle &\leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \\ (N-1) \left[ (\Delta J_k)^2 + (\Delta J_l)^2 \right] &\geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},\end{aligned}$$

where  $k, l, m$  take all the possible permutations of  $x, y, z$ .

# Derivation of the equations

- Criterion 2

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$

Proof: For product states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 = \sum_k (\Delta j_x^{(k)})^2 + (\Delta j_y^{(k)})^2 + (\Delta j_z^{(k)})^2 \geq \frac{N}{2}.$$

It is also true for separable states due to the convexity of separable states.

- Criterion 3

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2},$$

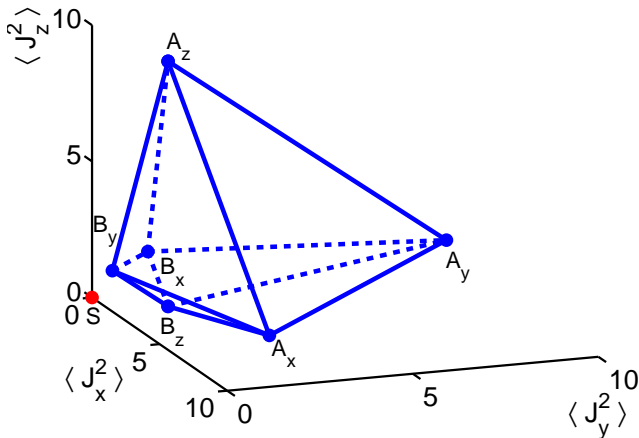
Proof: For product states

$$(N-1)(\Delta J_x)^2 + \frac{N}{2} - \langle J_y^2 \rangle - \langle J_z^2 \rangle = (N-1) \left( \frac{N}{4} - \frac{1}{4} \sum_k x_k^2 \right) - \frac{1}{4} \sum_{k \neq l} y_k y_l + z_k z_l = \dots \geq 0.$$

Here  $x_k = \langle \sigma_x^{(k)} \rangle$  and we have to use  $(\sum_k s_k)^2 \leq N \sum_k s_k$ .

# The polytope

- The previous inequalities, for fixed  $\langle J_{x/y/z} \rangle$ , describe a polytope in the  $\langle J_{x/y/z}^2 \rangle$  space. The polytope has six extreme points:  $A_{x/y/z}$  and  $B_{x/y/z}$ .
- For  $\langle \mathbf{J} \rangle = 0$  and  $N = 6$  the polytope is the following:





# The polytope II

- The coordinates of the extreme points are

$$A_x := \left[ \frac{N^2}{4} - \kappa(\langle J_y \rangle^2 + \langle J_z \rangle^2), \frac{N}{4} + \kappa \langle J_y \rangle^2, \frac{N}{4} + \kappa \langle J_z \rangle^2 \right],$$
$$B_x := \left[ \langle J_x \rangle^2 + \frac{\langle J_y \rangle^2 + \langle J_z \rangle^2}{N}, \frac{N}{4} + \kappa \langle J_y \rangle^2, \frac{N}{4} + \kappa \langle J_z \rangle^2 \right],$$

where  $\kappa := (N - 1)/N$ . The points  $A_{y/z}$  and  $B_{y/z}$  can be obtained from these by permuting the coordinates.

- Now it is easy to prove that an inequality is a necessary condition for separability: All the six points must satisfy it.

# The polytope III

- Let us take the  $\langle \mathbf{J} \rangle = 0$  case first.
- Then the state corresponding to  $A_x$  is the equal mixture of

$$|+1, +1, +1, +1, \dots\rangle_x$$

and

$$|-1, -1, -1, -1, \dots\rangle_x.$$

- The state corresponding to  $B_x$  is

$$|+1\rangle_x^{\otimes \frac{N}{2}} \otimes |-1\rangle_x^{\otimes \frac{N}{2}}.$$

- Separable states corresponding to  $A_{y/z}$  and  $B_{y/z}$  are defined similarly.

# The polytope IV

- General case:  $\langle \mathbf{J} \rangle \neq 0$ .
- A separable state corresponding to  $A_x$  is

$$\rho_{A_x} = p(|\psi_+\rangle\langle\psi_+|)^{\otimes N} + (1 - p)(|\psi_-\rangle\langle\psi_-|)^{\otimes N}.$$

Here  $|\psi_{+/-}\rangle$  are the single qubit states with Bloch vector coordinates  $(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = (\pm c_x, 2\langle J_y\rangle/N, 2\langle J_z\rangle/N)$  where  $c_x := \sqrt{1 - 4(\langle J_y\rangle^2 + \langle J_z\rangle^2)/N^2}$ . The mixing ratio is defined as  $p := 1/2 + \langle J_x\rangle/(Nc_x)$ .

- If  $N_1 := Np$  is an integer, we can also define the state corresponding to the point  $B_x$  as

$$|\phi_{B_x}\rangle = |\psi_+\rangle^{\otimes N_1} \otimes |\psi_-\rangle^{\otimes (N-N_1)}.$$

If  $N_1$  is not an integer then one can find a point  $B'_x$  such that such that its distance from  $B_x$  is smaller than  $\frac{1}{4}$ .

# In what sense is the characterization complete?

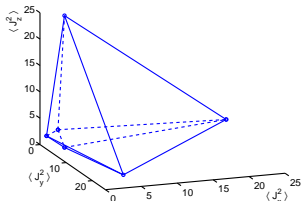
- For any value of  $\mathbf{J}$  there are separable states corresponding to  $A_{x/y/z}$ .
- For certain values of  $\mathbf{J}$  and  $N$  (e.g.,  $\mathbf{J} = 0$  and even  $N$ ) there are separable states corresponding to points  $B_{x/y/z}$ .
- However, there are always separable states corresponding to points  $B'_{x/y/z}$  such that their distance from  $B_{x/y/z}$  is smaller than  $\frac{1}{4}$ .
- In the limit  $N \rightarrow \infty$  for a fixed normalized angular momentum  $\frac{\mathbf{J}}{N/2}$  the difference between the volume of our polytope and the volume of set of points corresponding to separable states decreases with  $N$  as

$$\frac{\Delta V}{V} \propto N^{-2},$$

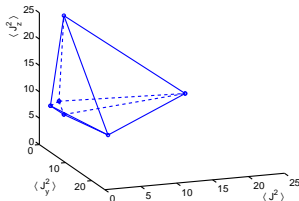
hence in the macroscopic limit the characterization is complete.

# Polytope for various values for J

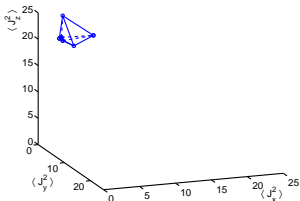
- The polytope for  $N = 10$  and  $J = (0, 0, 0)$ ,



$$J = (0, 0, 2.5),$$



and  $J = (0, 0, 4.5)$ .

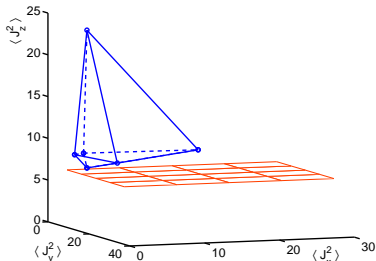


# Our inequalities vs. the standard spin squeezing criterion

- The standard spin squeezing criterion

$$\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2} \geq \frac{1}{N}$$

is satisfied by all points  $A_k$  and  $B_k$ , for  $B_z$  even equality holds.



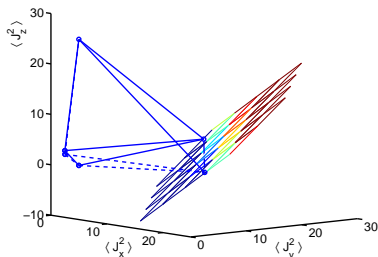
- Polytope for  $N = 10$  and  $J = (1.5, 0, 2.5)$ . States that are detected by the standard criterion are below the red plane.

# Our inequalities vs. the Korbicz-Cirac-Lewenstein inequalities

- For states with a separable two-qubit density matrix

$$\left( \langle J_k^2 \rangle + \langle J_l^2 \rangle - \frac{N}{2} \right)^2 + (N-1)^2 \langle J_m \rangle^2 \leq \langle J_m^2 \rangle + \frac{N(N-2)}{4}$$

holds. [J. Korbicz *et al.* PRL **95**, 120502 (2005).]



- Polytope for  $N = 10$  and  $J = (0, 0, 0)$ . States that are detected by the KCL criterion are below the plane. The plane contains two of the three  $A_k$  points.

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# Two-qubit entanglement

- Our criteria can detect entangled states for which the reduced two-qubit density matrix is separable.
- This might look surprising since all our criteria contain operator expectation values that can be computed knowing the average two-qubit density matrix

$$\rho_{12} := \frac{1}{N(N-1)} \sum_{k \neq l} \rho_{kl},$$

and no information on higher order correlation is used.

- Still, our criteria do not merely detect entanglement in the reduced two-qubit state!

## Two-qubit entanglement II

- Two-qubit symmetric separable states have the form

$$\rho_{12} = \sum_k p_k \rho_k \otimes \rho_k.$$

For such states it is always possible to find an  $N$ -qubit separable state, which has  $\rho_{12}$  as its reduced state:

$$\rho = \sum_k p_k \rho_k \otimes \rho_k \otimes \dots \otimes \rho_k.$$

Note the connection to the representability problem.

- However, there are two-qubit separable states for which this is not possible. For example, these can be of the form

$$\rho_{12} = \frac{1}{2}(\rho_1 \otimes \rho_2 + \rho_2 \otimes \rho_1).$$

Clearly, it is not easy to find an  $N$ -qubit state for such a state.

# Two-qubit entanglement III

- From the previous discussion it follows the following:  
For symmetric states, the violation of any entanglement criterion with  $\langle J_k \rangle$  and  $\langle J_k^2 \rangle$  implies the entanglement of the reduced two-qubit density matrix.
- This was found by Wang and Sanders for the standard spin-squeezing inequality.

[X. Wang and B.C. Sanders, PRA **68**, 012101 (2003).]

# Bound entanglement in spin chains

- Let us consider four spin-1/2 particles, interacting via the Hamiltonian

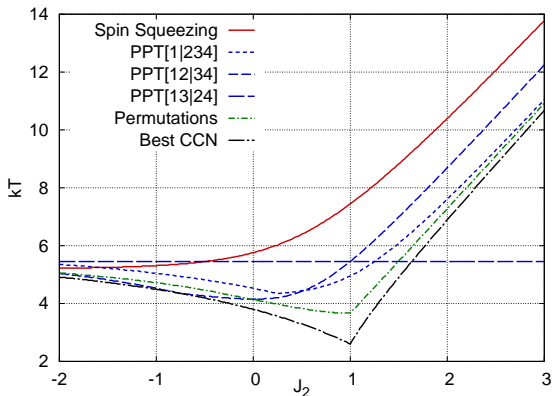
$$H = (h_{12} + h_{23} + h_{34} + h_{41}) + J_2(h_{13} + h_{24}),$$

where  $h_{ij} = \sigma_x^{(i)} \otimes \sigma_x^{(j)} + \sigma_y^{(i)} \otimes \sigma_y^{(j)} + \sigma_z^{(i)} \otimes \sigma_z^{(j)}$  is a Heisenberg interaction between the qubits  $i, j$ .

- For the above Hamiltonian we compute the thermal state  $\varrho(T, J_2) \propto \exp(-H/kT)$  and investigate its separability properties.
- For several separability criteria we calculate the maximal temperature, below which the criteria detect the states as entangled.

# Bound entanglement in spin chains II

- Bound temperatures for entanglement



For  $J_2 \gtrsim -0.5$ , the spin squeezing inequality is the strongest criterion for separability. It allows to detect entanglement even if the state has a positive partial transpose (PPT) with respect to **all** bipartition.

## Bound entanglement in spin chains III

- We found bound entanglement that is PPT with respect to all bipartitions in XY and Heisenberg chains, and also in XY and Heisenberg models on a completely connected graph, up to 10 qubits.
- Thus for these models, which appear in nature, there is a considerable temperature range in which the system is already PPT but not yet separable.

# Conclusions

- We presented a family of entanglement criteria that are able to detect any entangled state that can be detected based on the first and second moments of collective angular momenta.
- We explicitly determined the set of points corresponding to separable states in the space of first and second order moments.
- We applied our findings to examples of spin models, showing the presence of bound entanglement in these models.

\*\*\* THANK YOU \*\*\*