Detecting metrologically useful entanglement in Dicke states

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Introduction
- With the rapid development of quantum control it is now possible to create large scale entanglement in many physical systems, such as cold atoms or trapped ions.
- Entanglement conditions with collective measurements are important since in many quantum experiments the spins cannot be individually addressed.
- We discuss, how to detect multiparticle entanglement in Dicke states prepared in an experiment with few measurements.
- We also show how to verify the metrological usefulness of quantum states based on few measurements, without the need to carry out the metrological procedure itself.

Spin-squeezed states
- Entanglement criterion [4]
  \[ \mathcal{E}_n^q = N \left( \frac{(J_n^q)^2}{J_n^q} + J_n^q \right). \]
  If \( \mathcal{E}_n^q < 1 \) then the state is entangled.
- States detected are of the following type:
  **Variance of \( J_z \)** is small
  **J_{z\,large}**

Dicke states
- Dicke states are defined as
  \[ |D_{J,N}(\lambda)|^2 = \frac{1}{N!} \sum_{\lambda} \frac{\lambda!}{(\lambda - J)!^2} (\lambda - J)! (J)!^2. \]
- Spin-squeezed states: Measure \( \langle J_z \rangle \) to estimate the angle \( \theta \). (We cannot measure first moments, since they are zero.)
- For separable states
  \[ F_0[p,J] \leq N. \]
- For states with at most \( k \)-particle entanglement
  \[ F_0[p,J] \leq kN. \]

Entanglement depth
- Entanglement criterion for both Dicke states and spin-squeezed states [3,4].
  \[ (J_n^q)^2 \geq N \frac{J_n}{G_n(X)} \left( \frac{(J_n^q)^2}{J_n^q} + J_n^q \right). \]
  holds for states with an entanglement depth of at most \( k \) of an ensemble of \( N \) spin-1 particles. \( G_n(X) \) is a function obtained numerically.
  If a state violates the above criterion then it has at least an entanglement depth \( k + 1 \).

Bound QFI from below based on \( w_k = |\psi_k\rangle \).
- Using the Legendre transform technique, we arrive at the formula [3]
  \[ F_0[p,J] \geq \sup_{\{\lambda\}} \sup_{\mu} \sum_{k} r_k w_k - \sup_{\mu} \lambda_{\max} (M). \]
  where \( M = \sum r_k w_k - 4 \langle J_z - \mu \rangle^2. \)
- Our method works for systems with density matrices of size \( 1000 \times 1000 \) or even larger.
- Bounding the QFI for spin squeezing:

Quantum Fisher information
- Cramér-Rao bound on the precision of parameter estimation
  \[ (\Delta \theta)^2 \geq \frac{1}{F_0[p,A]}. \]
  where \( F_0[p,A] \) is the quantum Fisher information (QFI) defined as
  \[ F_0[p,A] = 2 \sum_{\lambda} \frac{(\lambda - \lambda_{\max})^2}{\lambda_{\max} A + \lambda_{\max} A} (\lambda A)^2, \]
  and \( \rho = \sum \lambda_{\lambda} |\lambda\rangle \langle \lambda| \).
- For separable states
  \[ F_0[p,J] \leq N. \]
- For states with at most \( k \)-particle entanglement
  \[ F_0[p,J] \leq kN. \]

Quantum metrology
- Experiments
  - Photonic systems with four and six qubits [1,5]
  - Bose Einstein condensates, thousands of atoms [2,6]

Spin-squeezed states
- Measure \( \langle J_z \rangle \) to estimate \( \theta \).
  - We cannot measure first moments, since they are zero.

Estimating the QFI
- Estimating the QFI II
- Related bibliography
  - Papers with our contributions
  - Literature