Tutorial: Quantum metrology from a quantum information science perspective
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Géza Tóth[1,2,3] and Iagoba Apellaniz[1] 

[1]Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary

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Outline

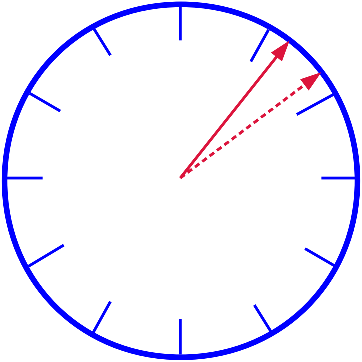
Why is quantum metrology interesting?

- Recent technological development has made it possible to realize large coherent quantum system, i.e., in cold gases.
- Can such quantum systems outperform classical systems?
- The problem can be understood better based on entanglement theory.

Outline

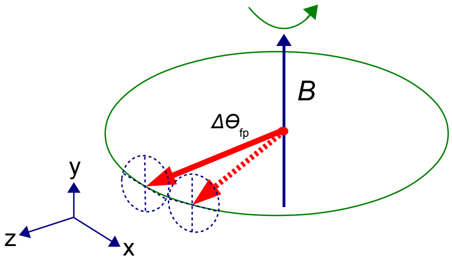
Estimating the angle of a clock arm

- Classical case: arbitrary precision ("in principle").



Magnetometry with the fully polarized state

- Let us see the quantum case.
- N spin-$1/2$ particles, all fully polarized in the z direction.
- Magtetic field B points to the y direction.



- Note the uncertainty ellipses. $\Delta\theta_{\rm fp}$ is the minimal angle difference we can measure.

Magnetometry with the fully polarized state II

- Collective angular momentum components

$$J_l := \sum_{n=1}^{N} j_l^{(n)}$$

for $l = x, y, z$, where $j_l^{(n)}$ are single particle operators.

- Dynamics

$$U_\theta = e^{-iJ_y\theta},$$

where $\hbar = 1$, and he angle θ is

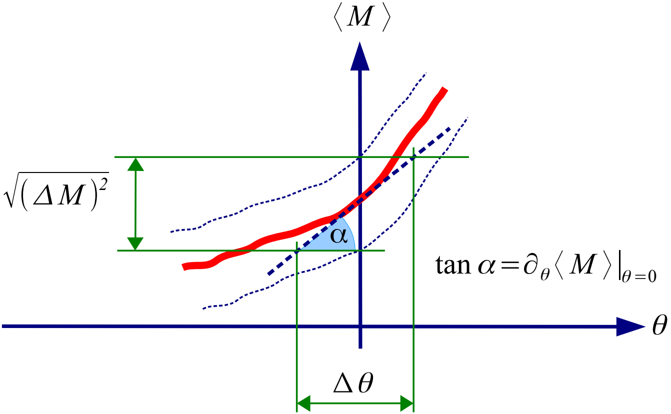
$$\theta = \gamma B t,$$

where γ is the gyromagnetic ratio, and t is the time.

Magnetometry with the fully polarized state IV

- Measure an operator M to get the estimate θ.
- The precision is given by the error propagation formula

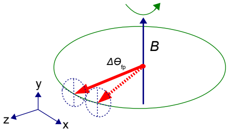
$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



Magnetometry with the fully polarized state V

- We measure the operator

$$M = J_x.$$



- Expectation value and variance

$$\begin{aligned}\langle M\rangle(\theta) &= \langle J_z\rangle\sin(\theta) + \langle J_x\rangle\cos(\theta),\\ (\Delta M)^2(\theta) &= (\Delta J_x)^2\cos^2(\theta) + (\Delta J_z)^2\sin^2(\theta)\\ &+ \left(\tfrac{1}{2}\langle J_xJ_z + J_zJ_x\rangle - \langle J_x\rangle\langle J_z\rangle\right)\sin(2\theta).\end{aligned}$$

- Using $\langle J_x\rangle = 0$, in the $\theta \to 0$ limit

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta\langle M\rangle|^2} = \frac{(\Delta J_x)^2}{\langle J_z\rangle^2} = \frac{1}{N}.$$

Magnetometry with the fully polarized state III

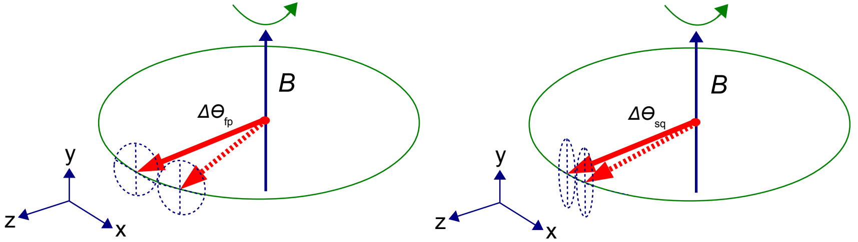
- It is not like a classical clock arm, we have a nonzero uncertainty

$$(\Delta\theta)^2 = \frac{1}{N}.$$

Outline

Magnetometry with the spin-squeezed state

- We can increase the precision by spin squeezing



fully polarized state (fp) spin-squeezed state (sq)

$\Delta\theta_{\mathrm{fp}}$ and $\Delta\theta_{\mathrm{sq}}$ are the minimal angle difference we can measure.

We can reach

$$(\Delta\theta)^2 < \frac{1}{N}.$$

Outline

Metrology with the GHZ state

- Greenberger-Horne-Zeilinger (GHZ) state

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}),$$

- Unitary

$$U_\theta = e^{-iJ_z\theta}.$$

- Dynamics

$$|\mathrm{GHZ}_N\rangle(t) = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + e^{-iN\theta}|1\rangle^{\otimes N}),$$

Metrology with the GHZ state II

- We measure

$$M = \sigma_x^{\otimes N},$$

which is the parity in the x-basis.

- Expectation value and variance

$$\langle M \rangle = \cos(N\theta), \qquad (\Delta M)^2 = \sin^2(N\theta).$$

- For $\theta \approx 0$, the precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \tfrac{1}{N^2}.$$

[e.g., photons: D. Bouwmeester, J. W. Pan, M. Daniell, H. Weinfurter and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999);

ions: C. Sackett et *al.*, Nature 404, 256 (2000).]

Metrology with the GHZ state III

- We reached the Heisenberg-limit

$$(\Delta\theta)^2 = \frac{1}{N^2}.$$

- The fully polarized state reached only the shot-noise limit

$$(\Delta\theta)^2 = \frac{1}{N}.$$

Outline

Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply "Dicke states" in the following) are defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*, Nature 2011]

Metrology with Dicke states

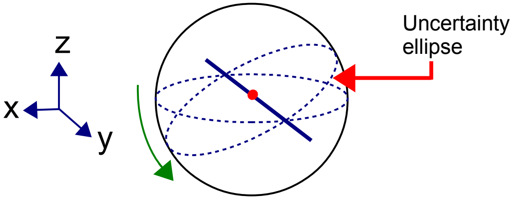
- For our symmetric Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ. (We cannot measure first moments, since they are zero.)



Metrology with Dicke states

- Dicke states are more robust to noise than GHZ states.
- Dicke states can also reach the Heisenberg-scaling like GHZ states.

[Metrology with cold gases: B. Lücke, M Scherer, J. Kruse, L. Pezze, F. Deuretzbacher, P. Hyllus, O. Topic, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, C. Klempt, Science 2011.]

[Metrology with photons: R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, P. Hyllus, L. Pezze, A. Smerzi, PRL 2011.]

Outline

Metrology with the singlet state

- For our singlet state

$$\langle J_l \rangle = 0, \quad \langle J_l^2 \rangle = 0, \quad l = x, y, z,$$

- Invariant under the actions of homogeneous magnetic fields, i.e., operations of the type $\exp(-iJ_{\vec{n}}\theta)$.
- Sensitive to gradients.
- We do not need to measure the homogeneous field, if we want to estimate the gradient.

[N. Behbood *et al.*, Phys. Rev. Lett. 113, 093601 (2014), covered in Scientific American "Quantum Entanglement Creates New State of Matter"; I. Urizar-Lanz *et al.*, PRA 2013.]

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

k-producibility/k-entanglement

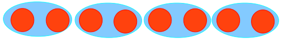
A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \ldots.$$

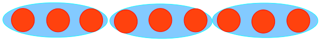
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., Gühne, GT, NJP 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

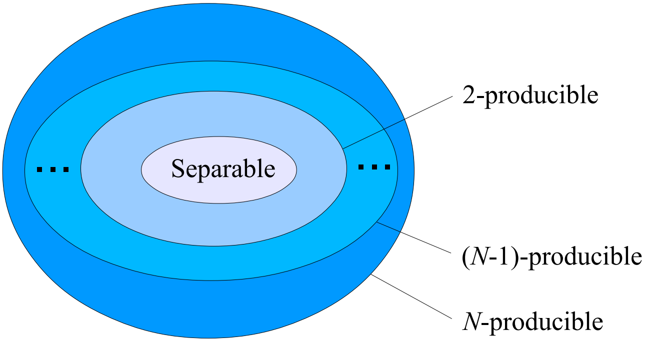


2-entangled



3-entangled

k-producibility/k-entanglement II



Outline

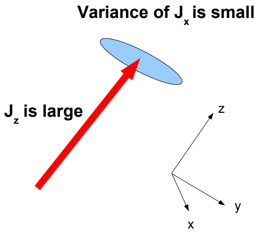
The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

$$\xi_s^2 = N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2}.$$

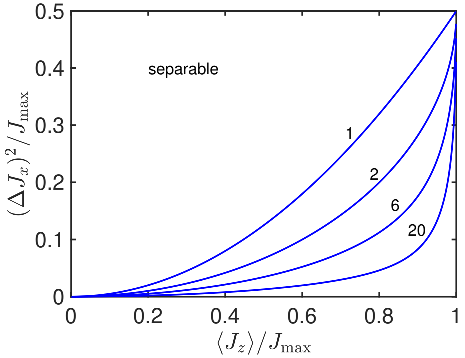
If $\xi_s^2 < 1$ then the state is entangled.
[Sørensen, Duan, Cirac, Zoller, Nature (2001).]

- States detected are like this:



Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).]

Generalized spin squeezing criteria for Dicke states

- The full set of entanglement criteria with collective observables has been obtained.
- One of these criteria is the following

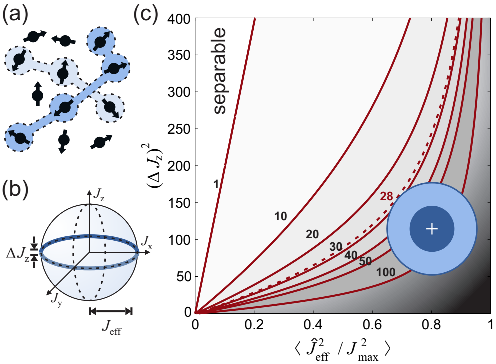
$$\langle J_x^2\rangle + \langle J_y^2\rangle \le (N-1)(\Delta J_z)^2 + \frac{N}{2}.$$

- It detects entanglement close to Dicke states.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

Multipartite entanglement detection around Dicke states

- Generalized spin squeezing inequality. BEC, 8000 particles. 28-particle entanglement is detected.



- $J^2_{\mathrm{eff}} = J_x^2 + J_y^2$ and $J_{\max} = N/2$.

[Lücke, Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, Phys. Rev. Lett. 112, 155304 (2014), also in Synopsys in physics.aps.org.]

Generalized spin squeezing criteria for singlet states

- As we have said, the full set of entanglement criteria with collective observables has been obtained.
- Another one of these criteria is the following

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}.$$

- It detects entanglement close to singlet states.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

Singlets

- For separable states of N spin-j particles

$$\xi^2_{\text{singlet}} = \frac{(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2}{Nj} \geq 1.$$

- For the singlet

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 = 0, \qquad \xi^2_{\text{singlet}} = 0.$$

- Number of particles entangled with the rest

$$N_{\text{e}} \geq N(1 - \xi^2_{\text{singlet}}).$$

[GT and M. W. Mitchell, New J. Phys 2010.]

[N. Behbood *et al.*, Phys. Rev. Lett. 113, 093601 (2014), covered in Scientific American "Quantum Entanglement Creates New State of Matter".]

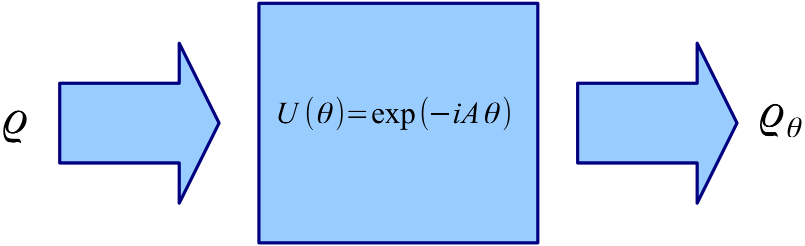
Our experience so far

- We looked at various setups.
- We find that better precision needs more entanglement.
- Question: Is this general?
- Answer: Yes.

Outline

Quantum metrology

- Fundamental task in metrology



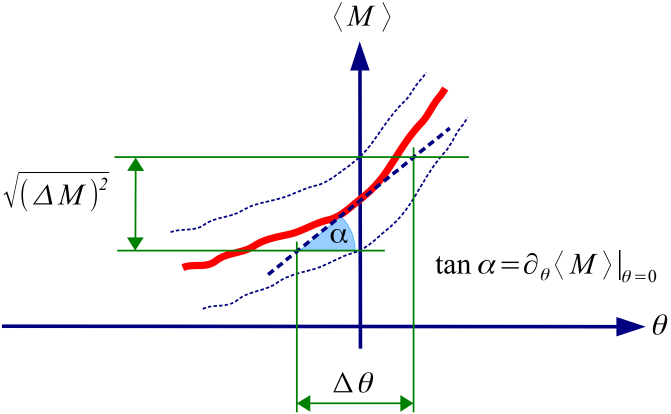
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation (slide repeated)

- Measure an operator M to get the estimate θ.
- The precision is given by the error propagation formula

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

Cramér-Rao bound on the precision of parameter estimation

For every M

$$(\Delta\theta)^2{}_M \geq \frac{1}{F_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The bound is even more general, includes any estimation strategy, even POVM's.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The optimal measurement

An optimal measurement can be carried out if we measure in the eigenbasis of the symmetric logarithmic derivative L given as

$$L = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|A|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- L is defined by

$$\frac{d\varrho_\theta}{d\theta} = \frac{1}{2}(L\varrho_\theta + \varrho_\theta L).$$

- Unitary dynamics with the Hamiltonian A

$$\frac{d\varrho_\theta}{d\theta} = i(\varrho_\theta A - A\varrho_\theta).$$

- Hence, the formula above can be obtained.
- Relation to the QFI: $F_Q[\varrho, A] = \mathrm{Tr}(L^2 \varrho)$.

Multi-parameter estimation

The Cramér-Rao bound for the multi-parameter case is

$$C - F^{-1} \geq 0.$$

- C is now the covariance matrix with elements

$$C_{mn} = \langle \theta_m \theta_n \rangle - \langle \theta_m \rangle \langle \theta_n \rangle.$$

- F is the Fisher matrix

$$F_{mn} \equiv F_Q[\varrho, A_m, A_n] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} \langle k|A_m|l \rangle \langle l|A_n|k \rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|.$

Quantum Fisher information and the fidelity

The quantum Fisher information appears in the Taylor expansion of F_B

$$F_B(\varrho, \varrho_\theta) = 1 - \theta^2 \frac{F_Q[\varrho, A]}{4} + \mathcal{O}(\theta^3),$$

where

$$\varrho_\theta = \exp(-iA\theta)\varrho\exp(+iA\theta).$$

- Bures fidelity

$$F_B(\varrho_1, \varrho_2) = \mathrm{Tr}\left(\sqrt{\sqrt{\varrho_1}\varrho_2\sqrt{\varrho_1}}\right)^2.$$

- Clearly,

$$0 \le F_B(\varrho_1, \varrho_2) \le 1.$$

The fidelity is 1 only if $\varrho_1 = \varrho_2$.

Convexity of the quantum Fisher information

- For pure states, it equals four times the variance,

$$F_Q[|\Psi\rangle, A] = 4(\Delta A)^2{}_\Psi.$$

- For mixed states, it is convex

$$F_Q[\varrho, A] \leq \sum_k p_k F_Q[|\Psi_k\rangle, A],$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

Outline

Magnetometry with a linear interferometer

- The Hamiltonian A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

  There are no interaction terms.

- The dynamics rotates all spins in the same way.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \le N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \le kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

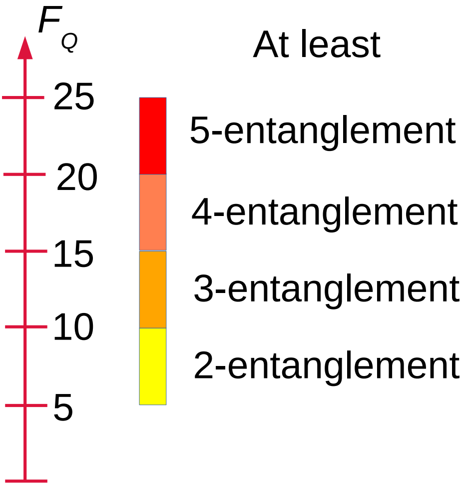
- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2,$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

The quantum Fisher information vs. entanglement

5 spin-1/2 particles



Let us use the Cramér-Rao bound

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{N}, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

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$$(\Delta\theta)^2 \propto \frac{1}{N^2},$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

Noisy metrology: Simple example

- A particle with a state ϱ_1 passes trough a map that turns its internal state to the fully mixed state with some probability p as

$$\epsilon_p(\varrho_1) = (1-p)\varrho_1 + p\tfrac{\mathbb{1}}{2}.$$

- This map acts in parallel on all the N particles

$$\epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^{N} p_n \varrho_n,$$

where the state obtained after n particles decohered into the completely mixed state is

$$\varrho_n = \tfrac{1}{N!}\sum_k \Pi_k \left[\left(\tfrac{\mathbb{1}}{2}\right)^{\otimes n} \otimes \mathrm{Tr}_{1,2,\ldots,n}(\varrho)\right] \Pi_k^\dagger.$$

The summation is over all permutations Π_k. The probabilities are

$$p_n = \binom{N}{n} p^n (1-p)^{(N-n)}.$$

Noisy metrology: Simple example II

- Rewriting it

$$\epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^{N} p_n \tfrac{1}{N!} \sum_k \Pi_k \left[\left(\tfrac{\mathbb{1}}{2}\right)^{\otimes n} \otimes \mathrm{Tr}_{1,2,\ldots,n}(\varrho) \right] \Pi_k^\dagger.$$

- For the noisy state

$$(\Delta J_x)^2 \geq \sum_n p_n (\Delta J_x)^2{}_{\varrho_n} \geq \sum_n p_n \tfrac{n}{4} = \tfrac{pN}{4}.$$

- Hence, for the precision shot-noise scaling follows

$$(\Delta\theta)^2 = \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} \geq \frac{\frac{pN}{4}}{\frac{N^2}{4}} \propto \frac{1}{N}.$$

Noisy metrology: General treatment

- In the most general case, uncorrelated single particle noise leads to shot-noise scaling after some particle number.

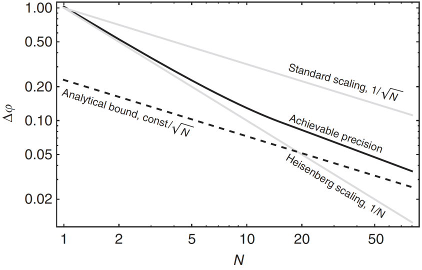

Figure from

[R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

- Correlated noise is different.

Reviews

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Summary

- We reviewed quantum metrology from a quantum information point of view.

See:

Géza Tóth and Iagoba Apellaniz,

Quantum metrology from a quantum information science perspective„

J. Phys. A: Math. Theor. 47, 424006 (2014),
special issue "50 years of Bell's theorem"
(open access).

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