Outline

1 Motivation
   - Why is quantum metrology interesting?

2 Simple examples of quantum metrology
   - Magnetometry with the fully polarized state
   - Magnetometry with the spin-squeezed state
   - Metrology with the GHZ state
   - Dicke states
   - Singlet states

3 Entanglement theory
   - Multipartite entanglement
   - The spin-squeezing criterion

4 Quantum metrology using the quantum Fisher information
   - Quantum Fisher information
   - Quantum Fisher information in linear interferometers
Why is quantum metrology interesting?

- Recent technological development has made it possible to realize large coherent quantum systems, i.e., in cold gases.

- Can such quantum systems outperform classical systems?

- The problem can be understood better based on entanglement theory.
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Estimating the angle of a clock arm

Classical case: arbitrary precision ("in principle").
Let us see the quantum case.

\( N \) spin-1/2 particles, all fully polarized in the \( z \) direction.

Magnetic field \( B \) points to the \( y \) direction.

Note the uncertainty ellipses. \( \Delta \theta_{fp} \) is the minimal angle difference we can measure.
Collective angular momentum components

\[ J_l := \sum_{n=1}^{N} j_l^{(n)} \]

for \( l = x, y, z \), where \( j_l^{(n)} \) are single particle operators.

Dynamics

\[ U_\theta = e^{-iJ_y \theta}, \]

where \( \hbar = 1 \), and the angle \( \theta \) is

\[ \theta = \gamma B t, \]

where \( \gamma \) is the gyromagnetic ratio, and \( t \) is the time.
Magnetometry with the fully polarized state IV

- Measure an operator $M$ to get the estimate $\theta$.

- The precision is given by the error propagation formula

\[ (\Delta \theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}. \]
Magnetometry with the fully polarized state V

- We measure the operator

\[ M = J_x. \]

- Expectation value and variance

\[
\langle M \rangle(\theta) = \langle J_z \rangle \sin(\theta) + \langle J_x \rangle \cos(\theta),
\]

\[
(\Delta M)^2(\theta) = (\Delta J_x)^2 \cos^2(\theta) + (\Delta J_z)^2 \sin^2(\theta)
\]

\[
+ \left( \frac{1}{2} \langle J_x J_z + J_z J_x \rangle - \langle J_x \rangle \langle J_z \rangle \right) \sin(2\theta).
\]

- Using \( \langle J_x \rangle = 0 \), in the \( \theta \to 0 \) limit

\[
(\Delta \theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} = \frac{1}{N}.
\]
It is not like a classical clock arm, we have a nonzero uncertainty

\[(\Delta \theta)^2 = \frac{1}{N}.\]
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   **Magnetometry with the spin-squeezed state**
   Metrology with the GHZ state
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Magnetometry with the spin-squeezed state

- We can increase the precision by spin squeezing

\[ \Delta \theta_{fp} \] and \[ \Delta \theta_{sq} \] are the minimal angle difference we can measure.

We can reach

\[ (\Delta \theta)^2 < \frac{1}{N}. \]
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Metrology with the GHZ state

- **Greenberger-Horne-Zeilinger (GHZ) state**

  \[ |\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes N + |1\rangle \otimes N), \]

- **Unitary**

  \[ U_\theta = e^{-iJ_z \theta}. \]

- **Dynamics**

  \[ |\text{GHZ}_N\rangle(t) = \frac{1}{\sqrt{2}} (|0\rangle \otimes N + e^{-iN\theta} |1\rangle \otimes N), \]
We measure

\[ M = \sigma_x^\otimes N, \]

which is the parity in the \( x \)-basis.

Expectation value and variance

\[ \langle M \rangle = \cos(N\theta), \quad (\Delta M)^2 = \sin^2(N\theta). \]

For \( \theta \approx 0 \), the precision is

\[ (\Delta \theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{N^2}. \]

[e.g., photons: D. Bouwmeester, J. W. Pan, M. Daniell, H. Weinfurter and A. Zeilinger, Phys. Rev. Lett. 82, 1345 (1999);
ions: C. Sackett et al., Nature 404, 256 (2000).]
We reached the Heisenberg-limit

\[(\Delta \theta)^2 = \frac{1}{N^2}.\]

The fully polarized state reached only the shot-noise limit

\[(\Delta \theta)^2 = \frac{1}{N}.\]
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Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply “Dicke states” in the following) are defined as

$$|D_N\rangle = \left(\frac{N}{\sqrt{2}}\right)^{-1/2} \sum_k P_k \left( |0\rangle^N \otimes |1\rangle^N \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left( |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$


For our symmetric Dicke state

\[ \langle J_l \rangle = 0, \ l = x, y, z, \ \langle J^2_z \rangle = 0, \ \langle J^2_x \rangle = \langle J^2_y \rangle = \text{large}. \]

Linear metrology

\[ U = \exp(-iJ_y \theta). \]

Measure \( \langle J^2_z \rangle \) to estimate \( \theta \). (We cannot measure first moments, since they are zero.)
Metrology with Dicke states

- Dicke states are more robust to noise than GHZ states.
- Dicke states can also reach the Heisenberg-scaling like GHZ states.


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Metrology with the singlet state

- For our singlet state

\[ \langle J_l \rangle = 0, \quad \langle J_l^2 \rangle = 0, \quad l = x, y, z, \]

- Invariant under the actions of homogeneous magnetic fields, i.e., operations of the type \( \exp(-iJ_{\vec{n}}\theta) \).

- Sensitive to gradients.

- We do not need to measure the homogeneous field, if we want to estimate the gradient.

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A state is *(fully) separable* if it can be written as

\[ \sum_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes ... \otimes \rho_k^{(N)}. \]

If a state is not separable then it is *entangled* (Werner, 1989).
A pure state is \textit{k-producible} if it can be written as

\[ |\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \ldots \]

where \(|\Phi_i\rangle\) are states of at most \(k\) qubits.

A mixed state is \(k\)-producible, if it is a mixture of \(k\)-producible pure states.

[ e.g., Gühne, GT, NJP 2005. ]

If a state is not \(k\)-producible, then it is at least \((k + 1)\)-particle entangled.

2-entangled

3-entangled
$k$-producibility/$k$-entanglement II

Separable

2-producible

(N-1)-producible

N-producible
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The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

\[ \xi_s^2 = N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2}. \]

If \( \xi_s^2 < 1 \) then the state is entangled.

[Sørensen, Duan, Cirac, Zoller, Nature (2001).]

- States detected are like this:
Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").

\[ N = 100 \text{ spin-1/2 particles, } J_{\text{max}} = \frac{N}{2}. \]

Generalized spin squeezing criteria for Dicke states

- The full set of entanglement criteria with collective observables has been obtained.

- One of these criteria is the following

  \[ \langle J_x^2 \rangle + \langle J_y^2 \rangle \leq (N - 1)(\Delta J_z)^2 + \frac{N}{2}. \]

- It detects entanglement close to Dicke states.

Multipartite entanglement detection around Dicke states

- Generalized spin squeezing inequality. BEC, 8000 particles. 28-particle entanglement is detected.

\[ \Delta J_z^2 \leq J_{\text{eff}}^2 J_z^2 J_x^2 J_y^2 \]

- \( J_{\text{eff}}^2 = J_x^2 + J_y^2 \) and \( J_{\text{max}} = N/2 \).

Generalized spin squeezing criteria for singlet states

- As we have said, the full set of entanglement criteria with collective observables has been obtained.

- Another one of these criteria is the following

\[(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}\]

- It detects entanglement close to singlet states.

Singlets

- For separable states of $N$ spin-$j$ particles

\[ \xi_{\text{singlet}}^2 = \frac{(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2}{Nj} \geq 1. \]

- For the singlet

\[ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 = 0, \quad \xi_{\text{singlet}}^2 = 0. \]

- Number of particles entangled with the rest

\[ N_e \geq N(1 - \xi_{\text{singlet}}^2). \]


Our experience so far

- We looked at various setups.

- We find that better precision needs more entanglement.

- Question: Is this general?

- Answer: Yes.
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Quantum metrology

- Fundamental task in metrology

\[ U(\theta) = \exp(-iA\theta) \]

- We have to estimate \( \theta \) in the dynamics

\[ U = \exp(-iA\theta). \]
Precision of parameter estimation (slide repeated)

- Measure an operator $M$ to get the estimate $\theta$.

- The precision is given by the error propagation formula

\[
(\Delta \theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.
\]
The quantum Fisher information

Cramér-Rao bound on the precision of parameter estimation

For every $M$

$$(\Delta \theta)^2_M \geq \frac{1}{F_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The bound is even more general, includes any estimation strategy, even POVM’s.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k| A| l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle \langle k|$. 
The optimal measurement

An optimal measurement can be carried out if we measure in the eigenbasis of the symmetric logarithmic derivative $L$ given as

$$L = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| |k\rangle\langle k| A|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|.$

- $L$ is defined by
  $$\frac{d\varrho_\theta}{d\theta} = \frac{1}{2} (L\varrho_\theta + \varrho_\theta L).$$

- Unitary dynamics with the Hamiltonian $A$
  $$\frac{d\varrho_\theta}{d\theta} = i(\varrho_\theta A - A\varrho_\theta).$$
  Hence, the formula above can be obtained.

- Relation to the QFI: $F_Q[\varrho, A] = \text{Tr}(L^2 \varrho).$
The Cramér-Rao bound for the multi-parameter case is

\[ C - F^{-1} \geq 0. \]

- \( C \) is now the covariance matrix with elements
  
  \[ C_{mn} = \langle \theta_m \theta_n \rangle - \langle \theta_m \rangle \langle \theta_n \rangle. \]

- \( F \) is the Fisher matrix
  
  \[ F_{mn} \equiv F_Q[\varrho, A_m, A_n] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} \langle k|A_m|l\rangle\langle l|A_n|k\rangle, \]

  where \( \varrho = \sum_k \lambda_k |k\rangle\langle k| \).
Quantum Fisher information and the fidelity

The quantum Fisher information appears in the Taylor expansion of $F_B$

$$F_B(\rho, \rho_\theta) = 1 - \theta^2 \frac{F_Q[\rho, A]}{4} + O(\theta^3),$$

where

$$\rho_\theta = \exp(-iA\theta)\rho \exp(+iA\theta).$$

- Bures fidelity

$$F_B(\rho_1, \rho_2) = \text{Tr} \left( \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right)^2.$$

- Clearly,

$$0 \leq F_B(\rho_1, \rho_2) \leq 1.$$

The fidelity is 1 only if $\rho_1 = \rho_2.$
Convexity of the quantum Fisher information

- For pure states, it equals four times the variance,

\[ F_Q[|\psi\rangle, A] = 4(\Delta A)^2_\psi. \]

- For mixed states, it is convex

\[ F_Q[\rho, A] \leq \sum_k p_k F_Q[|\psi_k\rangle, A], \]

where

\[ \rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|. \]
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The Hamiltonian $A$ is defined as

$$A = J_l = \sum_{n=1}^{N} j_{l}^{(n)}, \quad l \in \{x, y, z\}.$$ 

There are no interaction terms.

The dynamics rotates all spins in the same way.
The quantum Fisher information vs. entanglement

- For separable states
  \[ F_Q[\rho, J_l] \leq N, \quad l = x, y, z. \]

- For states with at most \( k \)-particle entanglement (\( k \) is divisor of \( N \))
  \[ F_Q[\rho, J_l] \leq kN. \]

- Macroscopic superpositions (e.g., GHZ states, Dicke states)
  \[ F_Q[\rho, J_l] \propto N^2, \]
The quantum Fisher information vs. entanglement

5 spin-1/2 particles

At least

5-entanglement

4-entanglement

3-entanglement

2-entanglement

$F_Q$
Let us use the Cramér-Rao bound

- For separable states

\[(\Delta \theta)^2 \geq \frac{1}{N}, \quad l = x, y, z.\]


- For states with at most $k$-particle entanglement ($k$ is divisor of $N$)

\[(\Delta \theta)^2 \geq \frac{1}{kN}.\]


- Macroscopic superpositions (e.g, GHZ states, Dicke states)

\[(\Delta \theta)^2 \propto \frac{1}{N^2},\]

Noisy metrology: Simple example

- A particle with a state $\varrho_1$ passes through a map that turns its internal state to the fully mixed state with some probability $p$ as

$$\epsilon_p(\varrho_1) = (1 - p)\varrho_1 + p\frac{1}{2}.$$ 

- This map acts in parallel on all the $N$ particles

$$\epsilon_p^\otimes N(\varrho) = \sum_{n=0}^{N} p_n \varrho_n,$$

where the state obtained after $n$ particles decohered into the completely mixed state is

$$\varrho_n = \frac{1}{N!} \sum_k \prod_k \left[ \left( \frac{1}{2} \right)^\otimes n \otimes \text{Tr}_{1,2,...,n}(\varrho) \right] \Pi_k^\dagger.$$ 

The summation is over all permutations $\Pi_k$. The probabilities are

$$p_n = \binom{N}{n} p^n (1 - p)^{(N-n)}.$$
Noisy metrology: Simple example II

Rewriting it

\[ \epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^{N} p_n \frac{1}{N!} \sum_k \Pi_k \left[ \left( \frac{1}{2} \right)^{\otimes n} \otimes \text{Tr}_{1,2,\ldots,n}(\varrho) \right] \Pi_k^\dagger. \]

For the noisy state

\[ (\Delta J_x)^2 \geq \sum_n p_n (\Delta J_x)^2_{\varrho_n} \geq \sum_n p_n \frac{n}{4} = \frac{pN}{4}. \]

Hence, for the precision shot-noise scaling follows

\[ (\Delta \theta)^2 = \frac{(\Delta J_x)^2}{\langle J_z \rangle^2} \geq \frac{pN}{N^2 \frac{4}{4}} \propto \frac{1}{N}. \]
In the most general case, uncorrelated single particle noise leads to shot-noise scaling after some particle number.

Figure from [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guță, Nature Comm. 2012.]

Correlated noise is different.
Reviews


Summary

- We reviewed quantum metrology from a quantum information point of view.

See:

Géza Tóth and Iagoba Apellaniz,
Quantum metrology from a quantum information science perspective,
special issue "50 years of Bell’s theorem"
(open access).

THANK YOU FOR YOUR ATTENTION!