Uncertainty relations with the variance and the quantum Fisher information

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Outline

1. **Motivation**
   - How can we improve uncertainty relations?
2. **Background**
   - Quantum Fisher information
   - Uncertainty relations
3. **Uncertainty relations with the variance and the QFI**
   - Uncertainty relations based on a convex roof of the bound
   - Uncertainty relations based on a concave roof of the bound
   - Several variances and the QFI

How can we improve uncertainty relations?

- There are many approaches to improve uncertainty relations.
- We show a method that replaces the variance with the quantum Fisher information in some well known uncertainty relations.
- We use convex/concave roofs over the decompostions of the density matrix.

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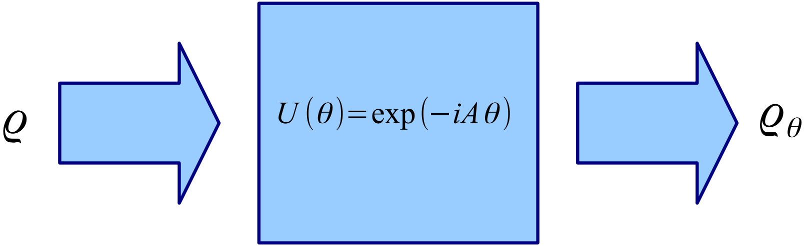
Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$

  where $F_Q[\varrho, A]$ is the quantum Fisher information, and m is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

  where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- See also convex roof over purifications.
  [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.]

Formula based on concave roofs

The variance is the concave roof of itself

$$(\Delta A)^2{}_{\varrho} = \max_{\{p_k, |\psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\psi_k},$$

where

$$\varrho = \sum_k p_k |\psi_k\rangle\langle\psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013);

GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014).

A single relation for the QFI and the variance

The previous statements can be concisely reformulated as follows. For any decomposition $\{p_k, |\psi_k\rangle\}$ of the density matrix ϱ we have

$$\frac{1}{4}F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2{}_{\psi_k} \leq (\Delta A)^2{}_{\varrho},$$

where the upper and the lower bounds are both tight.

- Note that

$$F_Q[\varrho, A] \leq 4(\Delta A)^2{}_{\varrho},$$

  where we have an equality for pure states.

- The QFI appears as a "pair" of variance.

The quantum Fisher information vs. entanglement

- For separable states of N spin-1/2 particles

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z, \qquad J_l = \sum_{n=1}^{N} j_l^{(n)}.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

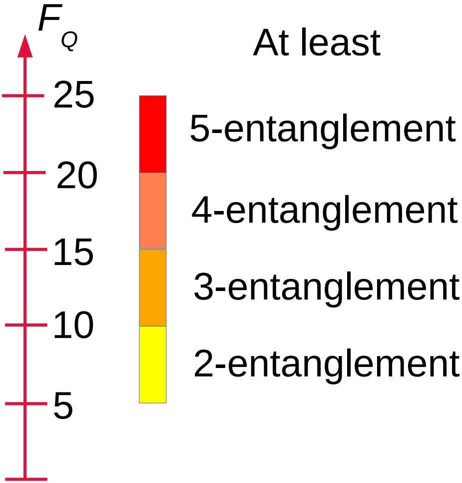
[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)]. $\rightarrow$ Many experiments with cold gases and photons.

- In general

$$F_Q[\varrho, J_l] \leq N^2.$$

The quantum Fisher information vs. entanglement II

5 spin-1/2 particles



Outline

Robertson-Schrödinger inequality

The Robertson-Schrödinger inequality is defined as

$$(\Delta A)^2{}_{\varrho}(\Delta B)^2{}_{\varrho} \geq \frac{1}{4}|L_{\varrho}|^2,$$

where the lower bound is given by

$$L_{\varrho} = \sqrt{|\langle\{A, B\}\rangle_{\varrho} - 2\langle A\rangle_{\varrho}\langle B\rangle_{\varrho}|^2 + |\langle C\rangle_{\varrho}|^2},$$

$\{A, B\} = AB + BA$ is the anticommutator, and we used the definition

$$C = i[A, B].$$

Important: L_{ϱ} is neither convex nor concave in ϱ.

Heisenberg uncertainty

The Heisenberg inequality is defined as

$$(\Delta A)^2{}_{\varrho}(\Delta B)^2{}_{\varrho} \geq \frac{1}{4}|\langle C\rangle_{\varrho}|^2,$$

where we used the definition

$$C = i[A, B].$$

The two inequalities together

We have two inequalities

$$(\Delta A)^2{}_\varrho (\Delta B)^2{}_\varrho \geq \frac{1}{4}|L_\varrho|^2 \geq \frac{1}{4}|\langle C \rangle_\varrho|^2.$$

The Heisenberg uncertainty can be saturated only if

$$|L_\varrho| = |\langle C \rangle_\varrho|.$$

Outline

Robertson-Schrödinger inequality for ϱ_k

- Consider a decomposition to mixed states

$$\varrho = \sum_k p_k \varrho_k.$$

- For such a decomposition, for all ϱ_k the Robertson-Schrödinger inequality holds

$$(\Delta A)^2{}_{\varrho_k} (\Delta B)^2{}_{\varrho_k} \geq \frac{1}{4} |L_{\varrho_k}|^2.$$

- Let us consider the inequality

$$\left(\sum_k p_k a_k\right) \left(\sum_k p_k b_k\right) \geq \left(\sum_k p_k \sqrt{a_k b_k}\right)^2,$$

where $a_k, b_k \geq 0$.

Uncertainty with the variance and the QFI

- Hence, we arrive at

$$\left[\sum_k p_k (\Delta A)^2{}_{\varrho_k}\right]\left[\sum_k p_k (\Delta B)^2{}_{\varrho_k}\right] \geq \frac{1}{4}\left[\sum_k p_k L_{\varrho_k}\right]^2.$$

- We can choose the decomposition such that

$$\sum_k p_k (\Delta B)^2{}_{\varrho_k} = F_Q[\varrho, B]/4.$$

- Due to the concavity of the variance we also know that

$$\sum_k p_k (\Delta A)^2{}_{\varrho_k} \leq (\Delta A)^2.$$

- Hence, it follows that

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\sum_k p_k L_{\varrho_k}\right)^2.$$

- In order to use the previous inequality, we need to know the decomposition $\{p_k, \varrho_k\}$ that minimizes $\sum_k p_k (\Delta B)^2{}_{\varrho_k}$.

Uncertainty with the variance and the QFI II

- We can have a inequality where we do not need to know that decomposition

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k, \varrho_k\}} \sum_k p_k L_{\varrho_k} \right)^2.$$

- On the right-hand side, the bound is defined based on a convex roof.
- It can be shown that we can move to pure state decompositions.
- We know that

$$L_{\psi_k} \geq |\langle C \rangle_{\psi_k}|$$

holds.

Uncertainty with the variance and the QFI III

- Then, we can obtain the inequality

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k |\langle C \rangle_{\psi_k}| \right)^2,$$

- Using

$$\sum_k p_k |\langle C \rangle_{\psi_k}| \geq \left| \sum_k p_k \langle C \rangle_{\psi_k} \right| \equiv |\langle C \rangle_{\varrho}|,$$

we arrive at the improved Heisenberg-Robertson uncertainty

$$(\Delta A)^2{}_{\varrho} F_Q[\varrho, B] \geq |\langle C \rangle_{\varrho}|^2.$$

Uncertainty with the variance and the QFI IV

- The Heisenberg uncertainty

$$(\Delta A)^2{}_{\varrho}(\Delta B)^2{}_{\varrho} \geq \frac{1}{4}|\langle i[A,B]\rangle_{\varrho}|^2.$$

- The improved Heisenberg uncertinty

$$(\Delta A)^2{}_{\varrho}F_Q[\varrho,B] \geq |\langle i[A,B]\rangle_{\varrho}|^2.$$

- It has been derived originally with a different method in

F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 92, 012102 (2015).

Conditions for saturation

- Conditions for saturating the relation with the simple bound

$$(\Delta A)^2_{\varrho} F_Q[\varrho, B] \geq \left(\min_{\{p_k, |\psi_k\rangle\}} \sum_k p_k L_{\psi_k} \right)^2 \geq |\langle C \rangle_{\varrho}|^2.$$

- We have to have equality on the right-hand side.
- Then, for all k, l we must have

$$\tfrac{1}{2}\langle\{A, B\}\rangle_{\psi_k} - \langle A \rangle_{\psi_k} \langle B \rangle_{\psi_k} = 0,$$

$$\begin{aligned} (\Delta A)^2_{\psi_k} &= (\Delta A)^2_{\psi_l}, \\ (\Delta B)^2_{\psi_k} &= (\Delta B)^2_{\psi_l}, \end{aligned}$$

$$|\langle C \rangle_{\psi_k}| = |\langle C \rangle_{\varrho}|,$$

etc.

Outline

Uncertainty relation based on a concave roof

- For any decomposition $\{p_k, \varrho_k\}$ we have

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\left(\sum_k p_k L_{\varrho_k}\right)^2,$$

where

$$L_\varrho = \sqrt{|\langle\{A, B\}\rangle_\varrho - 2\langle A\rangle_\varrho\langle B\rangle_\varrho|^2 + |\langle C\rangle_\varrho|^2}.$$

- We can even take a concave roof on the right-hand side

$${(\Delta A)^2}_\varrho{(\Delta B)^2}_\varrho \geq \frac{1}{4}\left(\max_{\{p_k,\varrho_k\}}\sum_k p_k L_{\varrho_k}\right)^2.$$

- We prove that for qubits the above inequality is saturated for all states.

Any decomposition leads to a valid bound

- A simple inequality that is valid

$$(\Delta A)^2{}_{\varrho}(\Delta B)^2{}_{\varrho} \geq \frac{1}{4}\left(\sum_k \lambda_k L_{|k\rangle}\right)^2,$$

  if we have an eigendecompostion

$$\varrho = \sum_k \lambda_k |k\rangle\langle k|.$$

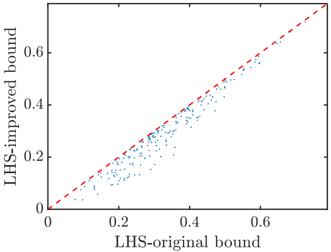
- We can even look for concave roof numerically.

Numerical example

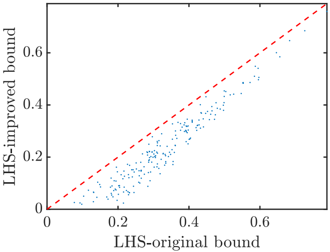
- For $d = 3$

$$(\Delta J_x)^2_{\varrho}(\Delta J_y)^2_{\varrho} \geq \frac{1}{4}\left(\max_{\{p_k,\varrho_k\}} \sum_k p_k L_{\varrho_k}\right)^2.$$

- Eigenvalues J_x and J_y are $-1, 0, +1$.



Using the eigendecomposition



Numerical search

Outline

Uncertainty relations with a variance and the QFI

- Similar ideas work even for a sum of two variances. For example, for a continuous variable system

$$(\Delta x)^2 + (\Delta p)^2 \geq 1$$

holds, where x and p are the position and momentum operators.

- Hence, for any decompositions of the density matrix it follows that

$$\sum_k p_k (\Delta x)^2{}_{\psi_k} + \sum_k p_k (\Delta p)^2{}_{\psi_k} \geq 1.$$

- For p we choose the decomposition that leads to the minimal value for the average variance, i.e., the QFI over four.

- Then, since $\sum_k p_k (\Delta x)^2{}_{\psi_k} \leq (\Delta x)^2$ holds, it follows that

$$(\Delta x)^2 + \frac{1}{4} F_Q[\varrho, p] \geq 1.$$

Uncertainty relations with two variances and the QFI

- Let us start from the relation for pure states

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq j,$$

where J_l are the spin components fulfilling

$$J_x^2 + J_y^2 + J_z^2 = j(j+1)\mathbb{1}.$$

- Based on similar ideas we arriving at

$$(\Delta J_x)^2 + (\Delta J_y)^2 + \frac{1}{4} F_Q[\varrho, J_z] \geq j.$$

- See parallel publication in

S.-H. Chiew and M. Gessner, Phys. Rev. Research 4, 013076 (2022).

Summary

- We showed how to derive new uncertainty relations with the variance and the quantum Fisher information based on simple convexity arguments.

See:

Géza Tóth and Florian Fröwis,

Uncertainty relations with the variance and the quantum Fisher information based on convex decompositions of density matrices,

Phys. Rev. Research 4, 013075 (2022).

THANK YOU FOR YOUR ATTENTION!

QUANTERA

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