

Quantum entanglement and its use in metrology

Géza Tóth

EHU Bilbao & HUN-REN Wigner Research Centre for Physics

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1 Motivation

- Motivation

2 Quantum entanglement

- Definition of entanglement
- Entanglement witnesses

3 Entanglement detection in multiparticle systems

- Detecting entanglement with the Hamiltonian of spin chains
- Entanglement detection close to Dicke states of few particles
- Entanglement detection in systems of very many particles

4 Quantum metrology

- Quantum metrology and entanglement

Motivation

- There have been many experiments recently aiming to create multiparticle quantum states.
- Quantum Information Science can help to find good targets for such experiments.
- **Highly entangled multiparticle quantum states** are good candidates for such experiments.
- Such states are needed in **quantum metrology**.

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Theory of quantum entanglement

- Statistical physics: the ground state of spin systems can be **factorizable**, i.e., product state

$$|\Psi_{12}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$$

or **not factorizable**. For product states $\langle A_1 B_2 \rangle = \langle A_1 \rangle \langle B_2 \rangle$.

- There are no pure states in an experiment. The product states must be generalized to the case of mixed states.
- **Separable states** = a mixture of product states.
- **Entangled (non-separable) states** are useful in certain quantum information processing applications.

Separable states

A state of N -particles is fully separable if it can be written in the following form

$$\rho_{\text{sep}} = \sum_m p_m \rho_m^{(1)} \otimes \rho_m^{(2)} \otimes \dots \otimes \rho_m^{(N)},$$

where $\rho_m^{(n)}$ are single-particle pure states.

- Separable states are essentially states that can be created without interaction between the particles by simply mixing the product states.

Separable states II

- Let us have two separable states

$$\varrho_{\text{sep},k} = \sum_m p_{m,k} \rho_{m,k}^{(1)} \otimes \rho_{m,k}^{(2)} \otimes \dots \otimes \rho_{m,k}^{(N)}$$

for $k = 1, 2$.

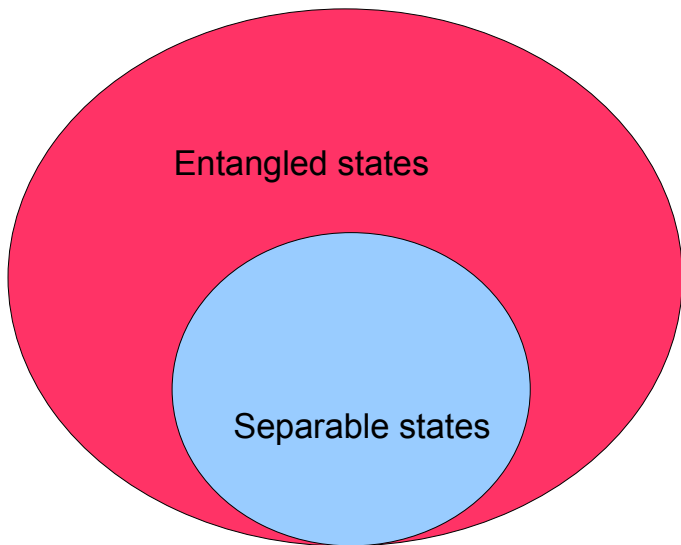
- Their mixture

$$\varrho = p \varrho_{\text{sep},1} + (1 - p) \varrho_{\text{sep},2},$$

where $0 \leq p \leq 1$, is also a separable state.

- Thus, the set of separable states is convex.

Convex sets



Many-body entanglement

A pure state **k -producible**, if it can be written as

$$|\Psi\rangle = \otimes_m |\psi_m\rangle,$$

where $|\psi_m\rangle$ are many-particle states with at most $k_m \leq k$ particles.

A mixed state is k -producible if it can be written as a mixture of pure k -producible states.

A state that is not k -producible, is at least **$(k + 1)$ -particle entangled**.

Genuine multipartite entanglement

- **Genuine multipartite entanglement**= N -particle entanglement in a system of N particles.
- **Biseparable states**=states that are not genuine multipartite entangled.
- For pure states, they are of the form

$$|\Psi_1\rangle \otimes |\Psi_2\rangle,$$

where $|\Psi_n\rangle$ are multiparticle states.

Examples

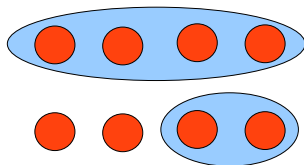
Examples

Two entangled states of four qubits:

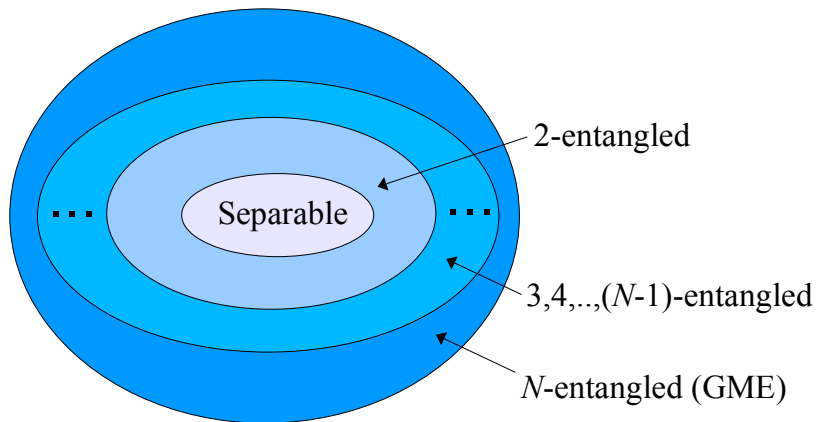
$$|\text{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

- The first state is genuine multipartite entangled, and 4-entangled.
- The second state is biseparable, and 2-entangled.



Convex sets



Convex sets of quantum states

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Entanglement detection

Looking at the definition

$$\rho_{\text{sep}} = \sum_m p_m \rho_m^{(1)} \otimes \rho_m^{(2)} \otimes \dots \otimes \rho_m^{(N)},$$

we see that is a difficult task to decide whether a quantum state is entangled or not.

There are no general methods.

Entanglement witness

An operator W is an **entanglement witness**, if the following conditions are fulfilled

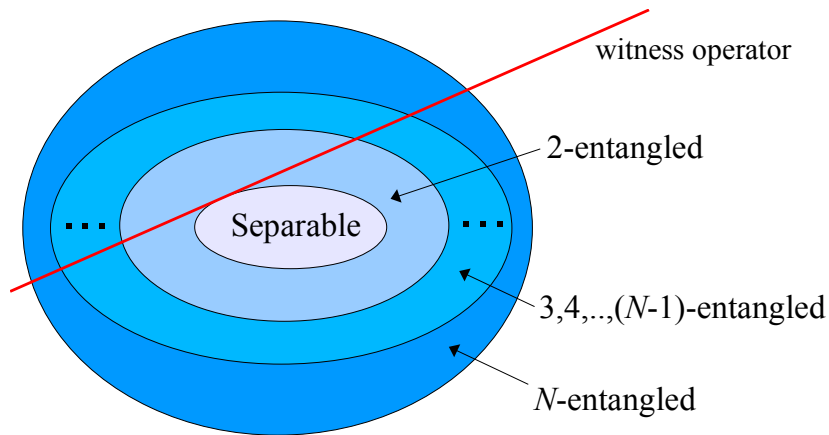
1. For separable states it is non-negative

$$\langle W \rangle_{\rho_{\text{sep}}} \equiv \text{Tr}(\rho_{\text{sep}} W) \geq 0.$$

2. There is such an entangled state ρ_{ent} for which the expectation value is negative

$$\langle W \rangle_{\rho_{\text{ent}}} < 0.$$

Entanglement witness II



Entanglement witness III

- We have to look for witness operators that are easy to measure.
- Or, if we consider a general witness operator, it might be a highly nonlocal operator.
- We cannot measure its expectation value directly.
- It must be decomposed it as

$$W = \sum_k A_k^{(1)} \otimes A_k^{(2)} \otimes \dots \otimes A_k^{(N)}.$$

- The expectation value can be obtained as

$$\langle W \rangle \equiv \text{Tr}(\rho W) = \sum_k \langle A_k^{(1)} \otimes A_k^{(2)} \otimes \dots \otimes A_k^{(N)} \rangle.$$

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Detecting entanglement with the Hamiltonian of spin chains

For a separable state of two qubits

$$-1 \leq \langle \sigma_x^{(1)} \sigma_x^{(2)} \rangle + \langle \sigma_y^{(1)} \sigma_y^{(2)} \rangle + \langle \sigma_z^{(1)} \sigma_z^{(2)} \rangle \leq 1.$$

holds.

- Basic idea: we take the maximum for product states of the type $|\psi_1\rangle \otimes |\psi_2\rangle$.

- For such states

$$\langle \sigma_l^{(1)} \sigma_l^{(2)} \rangle = \langle \sigma_l^{(1)} \rangle \langle \sigma_l^{(2)} \rangle$$

for $l = x, y, z$.

Detecting entanglement with the Hamiltonian of spin chains II

- The expectation value can be written as a scalar product

$$\langle \sigma_x^{(1)} \sigma_x^{(2)} \rangle + \langle \sigma_y^{(1)} \sigma_y^{(2)} \rangle + \langle \sigma_z^{(1)} \sigma_z^{(2)} \rangle = \vec{\mathcal{S}}_1 \cdot \vec{\mathcal{S}}_2,$$

where

$$\vec{\mathcal{S}}_n = \begin{pmatrix} \langle \sigma_x^{(n)} \rangle_{|\psi_n\rangle} \\ \langle \sigma_y^{(n)} \rangle_{|\psi_n\rangle} \\ \langle \sigma_z^{(n)} \rangle_{|\psi_n\rangle} \end{pmatrix}$$

for $n = 1, 2$.

- The Cauchy-Schwarz yields

$$|\vec{\mathcal{S}}_1 \vec{\mathcal{S}}_2| \leq |\vec{\mathcal{S}}_1| |\vec{\mathcal{S}}_2| = 1.$$

- The bound is also valid for separable states, since they are the mixture of product states.

Detecting entanglement with the Hamiltonian of spin chains III

- For the singlet state

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

we have

$$\langle \sigma_x^{(1)} \sigma_x^{(2)} \rangle + \langle \sigma_y^{(1)} \sigma_y^{(2)} \rangle + \langle \sigma_z^{(1)} \sigma_z^{(2)} \rangle = -3.$$

- An entanglement witness can be written as

$$W = \mathbb{1} + \sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)} + \sigma_z^{(1)} \sigma_z^{(2)}$$

for which

$$\langle W \rangle = -2.$$

Detecting entanglement with the Hamiltonian of spin chains IV

- The operator

$$W = \mathbb{1} + \sigma_x^{(1)}\sigma_x^{(2)} + \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_z^{(1)}\sigma_z^{(2)}$$

is a witness since

- (i) we have

$$\langle W \rangle \geq 0$$

for separable states,

- (ii) we have

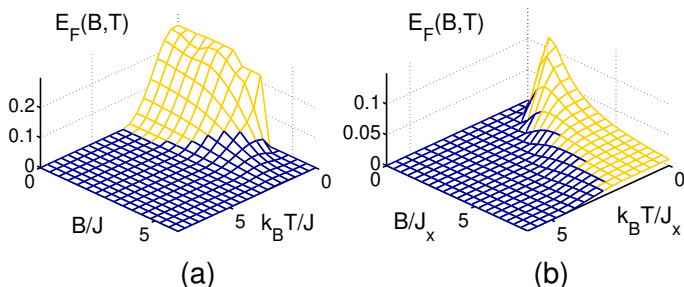
$$\langle W \rangle = -2$$

for the singlet state.

Spin chains

- If $\langle H \rangle$ is smaller than the energy minimum of the classical lattice, then the system is entangled.

Numerical results



(a) Heisenberg chain, 8 spins.

(b) Ising chain

E_F =entanglement of formation for the nearest neighbors

Yellow=detected states.

We detect the states that have more than minimal entanglement.

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Dicke states of N qubits

- Eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z ; $J_l := \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)}$.
- Symmetric Dicke states with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N^{(N/2)}\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

Due to symmetry, $\langle \vec{J}^2 \rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4^{(2)}\rangle = \frac{1}{\sqrt{6}} (|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, and H. Weinfurter, PRL 2007; Prevedel *et al.*, PRL 2007;
W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like Greenberger-Horne-Zelinger states (GHZ, \approx Schrödinger cat states)

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011.

GT, PRA 2012;

GT and Apellaniz, J. Phys. A, special issue for "50 year of Bell's theorem", 2014.

Entanglement detection close to Dicke states

For biseparable (not genuine multipartite entangled) ρ

$$F_{D_N} = \text{Tr}(\rho |D_N\rangle\langle D_N|) \leq \frac{1}{2} \frac{N}{N-1} =: C.$$

Any state that violates the above inequality has true multibody entanglement. (We omit the $N/2$ superscript.)

- For large N , $C \approx 1/2$. That is, only a fidelity of $1/2$ is required for a successful experiment.
- The limit cannot be smaller than $1/2$.
- Previously, it was only known for GHZ and cluster states that this limit was $1/2$.

Decomposition of the projector I

- The fidelity with respect to the Dicke state is

$$F_{D_N} = \text{Tr}(\rho |D_N\rangle\langle D_N|).$$

- The witness is

$$W = \frac{1}{2} \frac{N}{N-1} \mathbb{1} - |D_N\rangle\langle D_N|.$$

- We have to decompose the projector

$$|D_N\rangle\langle D_N| = \sum_k A_k \otimes A_k \otimes \dots \otimes A_k.$$

- We did this for $N = 6$, for which

$$W = 0.6\mathbb{1} - |D_N\rangle\langle D_N|.$$

Decomposition of the projector II

$$\begin{aligned} 64|D_6^{(3)}\rangle\langle D_6^{(3)}| &= -0.6[\mathbb{1}] + 0.3[x \pm \mathbb{1}] - 0.6[x] + 0.3[y \pm \mathbb{1}] - 0.6[y] + 0.2[z \pm \mathbb{1}] - 0.2[z] \\ &\quad + 0.2\text{Mermin}_{0,z} + 0.05[x \pm y \pm \mathbb{1}] - 0.05[x \pm z \pm \mathbb{1}] - 0.05[y \pm z \pm \mathbb{1}] \\ &\quad - 0.05[x \pm y \pm z] + 0.2[x \pm z] + 0.2[y \pm z] + 0.1[x \pm y] \\ &\quad + 0.6\text{Mermin}_{x,z} + 0.6\text{Mermin}_{y,z}. \end{aligned} \quad (31)$$

Here we use the notation $[x + y] = (\sigma_x + \sigma_y)^{\otimes 6}$, $[x + y + \mathbb{1}] = (\sigma_x + \sigma_y + \mathbb{1})^{\otimes 6}$, etc. The \pm sign denotes a summation over the two signs, i.e., $[x \pm y] = [x + y] + [x - y]$. The Mermin operators are defined as

$$\text{Mermin}_{a,b} := \sum_{k \text{ even}} (-1)^{k/2} \sum_k \mathcal{P}_k (\otimes_{i=1}^k \sigma_a \otimes_{i=k+1}^N \sigma_b), \quad (32)$$

where $\sigma_0 = \mathbb{1}$. That is, it is the sum of terms with even number of σ_a 's and σ_b 's, with the sign of the terms depending on the number of σ_a 's. The expectation value of the operators $\text{Mermin}_{a,b}$ can be measured based on the decomposition [23]

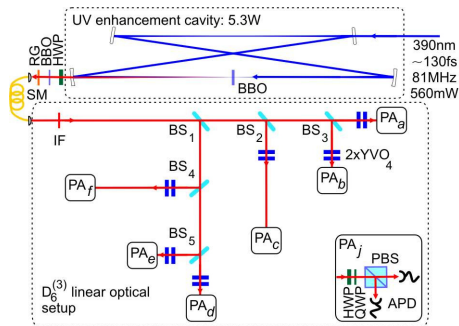
$$\text{Mermin}_{a,b} = \frac{2^{N-1}}{N} \sum_{k=1}^N (-1)^k \left[\cos\left(\frac{k\pi}{N}\right) a + \sin\left(\frac{k\pi}{N}\right) b \right]^{\otimes N}. \quad (33)$$

Results

settings [15,16]. We have determined $F_{D_6^{(3)}} = 0.654 \pm 0.024$ with a measurement time of 31.5 h. This allows the application of the generic entanglement witness [10] $\langle \mathcal{W}_g \rangle = 0.6 - F_{D_6^{(3)}} = -0.054 \pm 0.024$ and thus proves genuine six-qubit entanglement of the observed state with a significance of 2 standard deviations (Fig. 4).

W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, and H. Weinfurter
Experimental entanglement of a six-photon symmetric Dicke state, PRL 2009.

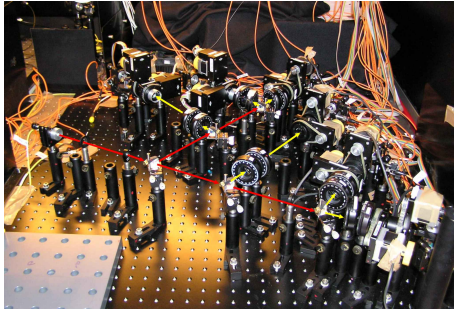
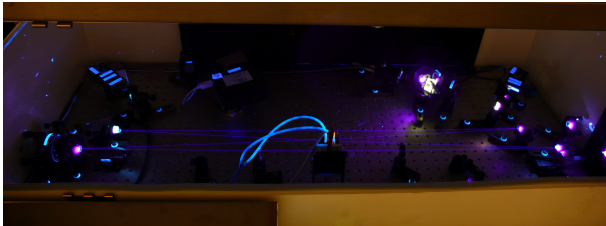
Experiments with photons



MPQ, München, experiment with six photons

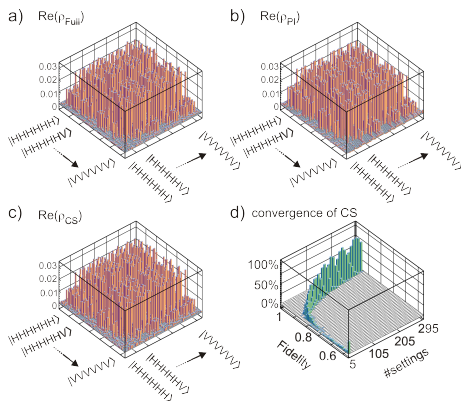
$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{20}} (|111000\rangle + |110100\rangle + \dots + |000111\rangle).$$

Experiments with photons



Experiments with photons

State tomography of the six-photon state



C. Schwemmer, G. Tóth, A. Niggebaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, Efficient Tomographic Analysis of a Six Photon State, PRL 2014.

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Many-particle systems

- For spin- $\frac{1}{2}$ particles, we can measure **expectation value** of the **collective** angular momentum operators:

$$\mathbf{J}_l := \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

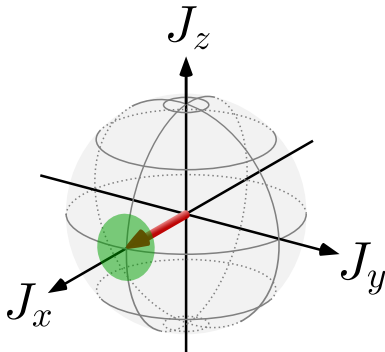
$$(\Delta J_l)^2 := \langle \mathbf{J}_l^2 \rangle - \langle \mathbf{J}_l \rangle^2$$

variances.

Fully polarized state

- State fully polarized in the x-direction

$$| + 1/2 \rangle_x^{\otimes N}.$$



We thank I. Appelaniz for the figure.

Spin squeezing

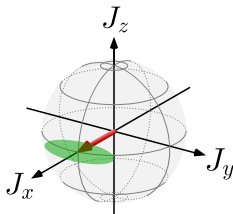
Definition

Uncertainty relation for the spin coordinates

$$(\Delta J_y)^2(\Delta J_z)^2 \geq \frac{1}{4}|\langle J_x \rangle|^2.$$

If $(\Delta J_z)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_x \rangle|$ then the state is called **spin squeezed** (mean spin in the x direction!).

M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).



We thank I. Appellaniz for the figure.

Spin squeezing II

Definition

Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

- Application: Quantum metrology, magnetometry. Used many times in experiments.

A. Sørensen *et al.*, Nature **409**, 63 (2001); experiments by E. Polzik, M.W. Mitchell with cold atomic ensembles; M. Oberthaler, Ph. Treutlein with Bose-Einstein condensates.

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

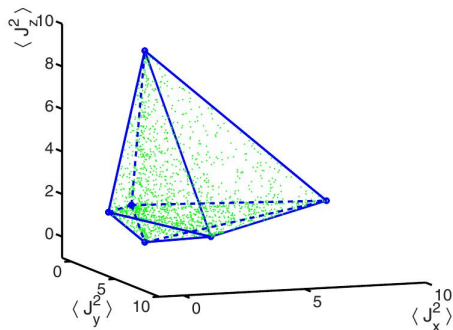
- Then any state violating the following inequalities is entangled:

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
$$(N-1) \left[(\Delta J_k)^2 + (\Delta J_l)^2 \right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z .

Generalized spin squeezing criteria for $j = \frac{1}{2} \mathbb{I}$

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality. For separable states

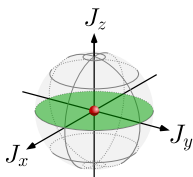
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

holds.

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2} \left(\frac{N}{2} + 1 \right) = \max.,$$
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.





**Bose-Einstein condensate
people**

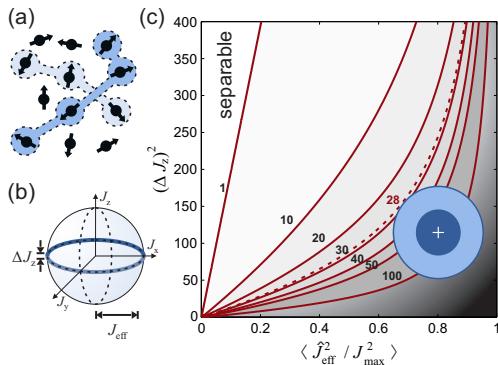


**Netflix movie
“Spectral”**

**Filmed in
Budapest**

Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J_{\text{eff}}^2 = J_x^2 + J_y^2, \quad J_{\text{max}} = \frac{N}{2}.$$

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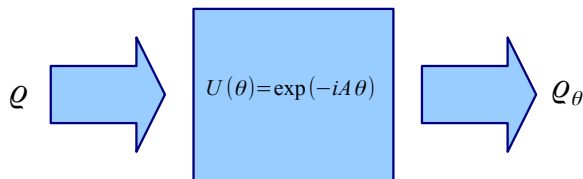
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Multipartite entanglement and quantum metrology



- Cramér-Rao bound

$$(\Delta\theta)^2 \geq \frac{1}{\nu F_Q[\rho, A]},$$

where ν is the number of independent repetitions.

- Quantum Fisher information

$$F_Q[\rho, A] = 2 \sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |A_{ij}|^2.$$

- Here λ_i denotes the eigenvalues of the density matrix, A_{ij} are the matrix elements of A in the eigenbasis of the density matrix.

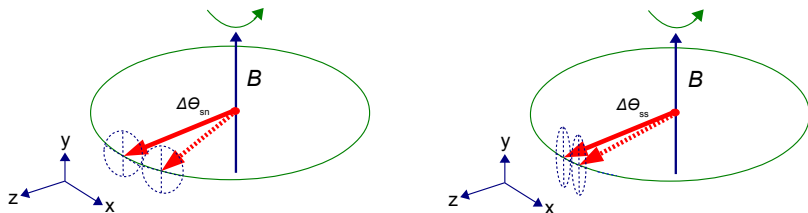
Special case $A = J_l$: linear interferometer

- The operator A is defined as

$$A = J_l = \sum_{n=1}^N j_l^{(n)}, \quad l \in \{x, y, z\}.$$

Note that there is **no interaction** between the particles.

- Magnetometry with a linear interferometer



The quantum Fisher information vs. entanglement

- **Shot-noise limit:** For separable states

$$F_Q[\varrho, J_l] \leq N, \quad (\Delta\theta)^2 \geq \frac{1}{\nu N}, \quad l = x, y, z.$$

Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010).

- **Heisenberg limit:** For entangled states

$$F_Q[\varrho, J_l] \leq N^2, \quad (\Delta\theta)^2 \geq \frac{1}{\nu N^2}, \quad l = x, y, z.$$

where the bound can be saturated.

Multipartite entanglement and Quantum Fisher information

For N -qubit k -producible states, the quantum Fisher information is bounded from above by

$$F_Q[\varrho, J_I] \leq sk^2 + (N - sk)^2,$$

where

$$s = \lfloor \frac{N}{k} \rfloor,$$

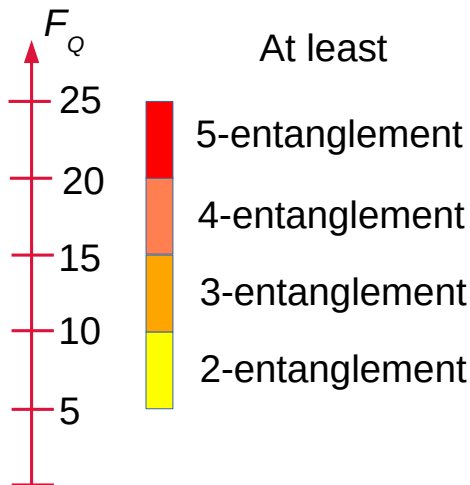
and $\lfloor \frac{N}{k} \rfloor$ denotes the integer part of $\frac{N}{k}$.

Simpler form with a bound that is not optimal

$$F_Q[\varrho, J_I] \leq Nk, \quad (\Delta\theta)^2 \geq \frac{1}{\nu Nk}.$$

Multipartite entanglement and Quantum Fisher information II

5 spin-1/2 particles



Metrology with Dicke states

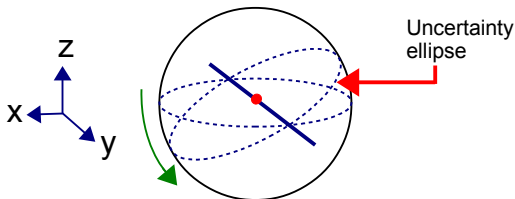
- For our symmetric Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- **Measure $\langle J_z^2 \rangle$ to estimate θ .** (We cannot measure first moments, since they are zero.)



Metrology with Dicke states

- They found that

$$F_Q[\varrho, J_I] \leq N,$$

is violated since they measured that

$$(\Delta\theta)^2 < \frac{1}{\nu N}.$$

Metrology with cold gases: B. Lücke, M Scherer, J. Kruse, L. Pezze, F. Deuretzbacher, P. Hyllus, O. Topic, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, C. Klempt, Science 2011.

Metrology with photons: R. Krschek, C. Schwemmer, W. Wieczorek, H. Weinfurter, P. Hyllus, L. Pezze, A. Smerzi, PRL 2011.

Summary

- Effective entanglement detection is very important in an experiment.
- We have presented methods that can be used in systems where the particles are addressable separately.
- We also presented methods that detect entanglement in multiparticle systems.
- We also found entanglement criteria based on quantum metrology.

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THANK YOU FOR YOUR ATTENTION!

