Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles
Quantum 7, 914 (2023)

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Outline

1. **Motivation**
   - Why entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Detecting multipartite entanglement of Dicke states**
   - Dicke state realized with a BEC of two-state atoms
4. **Detecting bipartite entanglement of Dicke states**
   - Bipartite entanglement from multipartite entanglement in BEC
   - Creating Dicke states in BEC
   - Entanglement detection in Dicke states
   - Our criteria work under realistic conditions

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea.
- In many cases, we need to detect bipartite entanglement.

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Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

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Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

  where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

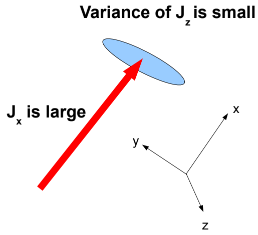
The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_s^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

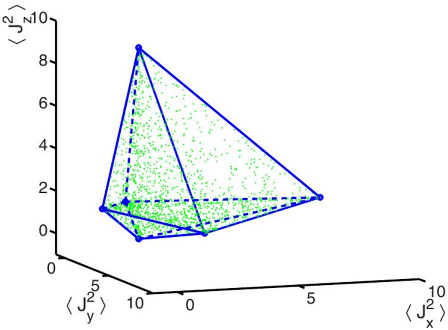
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \le \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \ge \frac{N}{2}, \qquad \text{(singlet states)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \le (N-1)(\Delta J_m)^2 + \frac{N}{2}, \qquad \text{(Dicke states)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \ge \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

singlets: GT, Phys. Rev. A 69, 052327 (2004);
all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Outline

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.
- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- Summing over all permutations.
- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

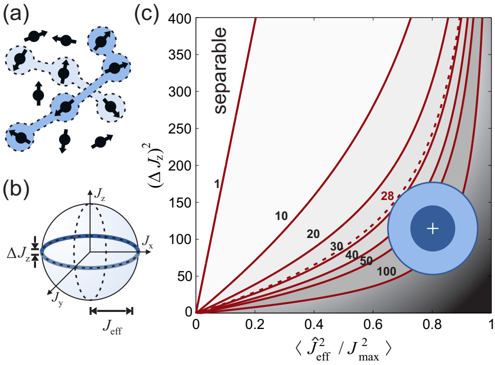
Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;
GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.

- ... are macroscopically entangled, like GHZ states.

Fröwis, Dür, PRL 2011.

Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J^2_{\mathrm{eff}} = J_x^2 + J_y^2, \quad J_{\mathrm{max}} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014).

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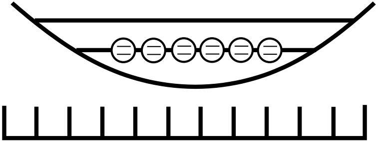
Bipartite entanglement from bosonic multipartite entanglement

- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

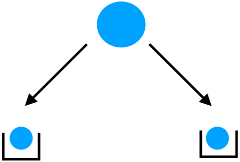
Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument



See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)

$|n_0 = 1\rangle |n_1 = 1\rangle$

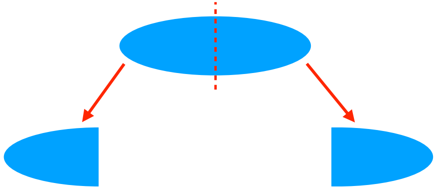


$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.

- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

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Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.

- Initially all atoms in the spin state $|j_z = 0\rangle$.

- Dynamics
$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$
Tunneling from mode 0 to the mode $+1$ and -1.

- Two-particle example:
$$|j_z = 0\rangle|j_z = 0\rangle \rightarrow \frac{1}{\sqrt{2}}(|j_z = +1\rangle|j_z = -1\rangle + |j_z = -1\rangle|j_z = +1\rangle)$$
$$= \text{Dicke state of 2 particles.}$$

Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- That is, $N - 2n$ particles remained in the $|j_z = 0\rangle$ state, while $2n$ particles form a symmetric Dicke state given as

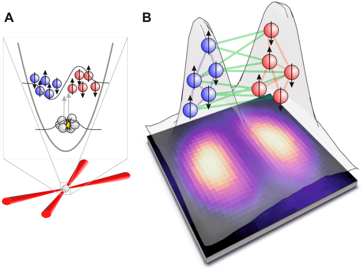
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}}\right),$$

  where we use $|0\rangle$ and $|1\rangle$ instead of $|j_z = -1\rangle$ and $|j_z = +1\rangle$.

- Half of the atoms in state $|0\rangle$, half of the atoms in state $|1\rangle$ + symmerization.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



[K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).]

Correlations for Dicke states

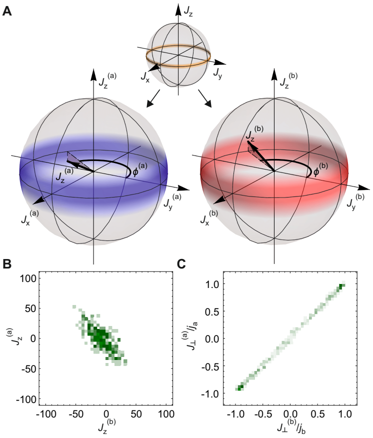
- For the Dicke state

$$\begin{aligned}(\Delta(J_x^a - J_x^b))^2 &\approx 0,\\ (\Delta(J_y^a - J_y^b))^2 &\approx 0,\\ (\Delta J_z)^2 = (\Delta(J_z^a + J_z^b))^2 &= 0.\end{aligned}$$

- Measurement results on well "*b*" can be predicted from measurements on "*a*"

$$\begin{aligned}J_x^b &\approx J_x^a, && \text{(correlation)}\\ J_y^b &\approx J_y^a, && \text{(correlation)}\\ J_z^b &= -J_z^a. && \text{(anti-correlation)}\end{aligned}$$

Correlations for Dicke states - experimental results



Here, $J_\perp^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

Outline

Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x\rangle^2 + \left\langle J_y\right\rangle^2).$$

Then,

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4}[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2\right\rangle + \left\langle J_y^2\right\rangle).$$

- Simple algebra yields

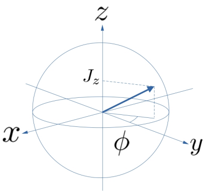
$$\left[(\Delta J_z)^2 + \frac{1}{4}\right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2\rangle + \langle J_y^2\rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2\rangle$ appears, not $\langle J_x\rangle^2$.

Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[(\Delta J_z)^2 + \frac{1}{4}\right]}_{\sim\text{ fluctuation of } J_z} \times \underbrace{\frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2\rangle + \langle J_y^2\rangle}}_{\sim\text{ phase fluctuation}} \geq \frac{1}{4}.$$



Handwaving description:
J_z and ϕ cannot be defined both with high accuracy.

Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_x^n = \frac{J_x^n}{\sqrt{j_n(j_n+1)}}, \quad \mathcal{J}_y^n = \frac{J_y^n}{\sqrt{j_n(j_n+1)}},$$

  where $n = a, b$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables $\rightarrow$ resistance to experimental imperfections.

Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x)^2 + (\Delta \mathcal{J}_y)^2\right] \geq \frac{1}{4}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle .$$

The two-well EPR-Steering criterion

Main result I

For states with a hidden state model,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{4}\left\langle (\mathcal{J}_x^a)^2 + (\mathcal{J}_y^a)^2 \right\rangle^2$$

holds.

Any state violating the inequality cannot be described by a hidden state model, i.e., the state is *steerable*.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_x^- &= \mathcal{J}_x^a - \mathcal{J}_x^b, \\ \mathcal{J}_y^- &= \mathcal{J}_y^a - \mathcal{J}_y^b. \end{aligned}$$

The two-well entanglement criterion

Main result II

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{16}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle^2$$

holds.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_x^- &= \mathcal{J}_x^a - \mathcal{J}_x^b, \\ \mathcal{J}_y^- &= \mathcal{J}_y^a - \mathcal{J}_y^b. \end{aligned}$$

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Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.

- Postselecting for a given particle number is not feasible.

- Hence, the density matrix is of the form

$$\varrho = \sum_{j_a,j_b} Q_{j_a,j_b} \varrho_{j_a,j_b},$$

  where Q_{j_a,j_b} are probabilities and ϱ_{j_a,j_b} are states.

- Here $j_a = N_a/2,\ j_b = N_b/2$.

- ϱ is entangled iff at least one of the ϱ_{j_a,j_b} is entangled.

Problem 1: Varying particle number II

- Splitting noise: Even if we have a constant total particle number, the ensemble will not be evenly split.

- Probability distribution for having $N/2 + x$ particles

$$p_x = 2^{-N}\binom{N}{N/2 + x}.$$

- Variance

$$\mathrm{var}(N_a) = \mathrm{var}(x) = \langle x^2 \rangle = \frac{N}{4}.$$

- Collective variance

$$[\Delta(J_l^a - J_l^b)]^2 \approx \sum_{x=-N/2}^{N/2} p_x \left(\frac{N}{8} + \frac{1}{2}x^2\right) = \frac{N}{8} + \frac{1}{2}\mathrm{var}(x) = \frac{N}{4}.$$

Twice as large due to the unequal splitting.

Problem 1: Varying particle number III

- $N/2 : N/2$ splitting:

$$[\Delta(J_l^a - J_l^b)]^2 = \frac{N}{8}.$$

- Real splitting with partition noise:

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}.$$

Problem 1: Varying particle number IV

- Solution: Let use the normalized quantity mentioned before

$$\mathcal{J}_l^- = \frac{1}{\sqrt{j_a(j_a+1)}} J_l^a - \frac{1}{\sqrt{j_b(j_b+1)}} J_l^b$$

for $l = x, y$.

- We obtain

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{N}{N^2/2 + 4N - 2x^2}.$$

- After splitting $|x| \lesssim \sqrt{N/4}$.

- We have

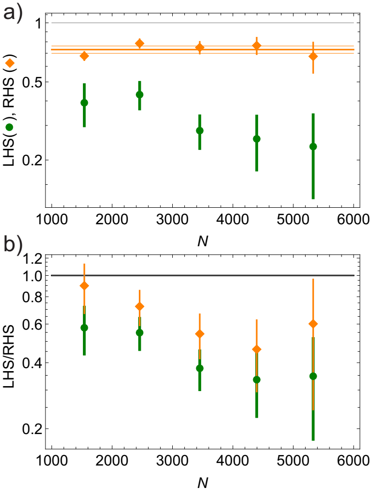
$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{2}{N}.$$

$(\Delta \mathcal{J}_l)^2$ is not sensitive to the fluctuation of x if N is large.

Problem 2: States are not always symmetric in a BEC of two-state atoms

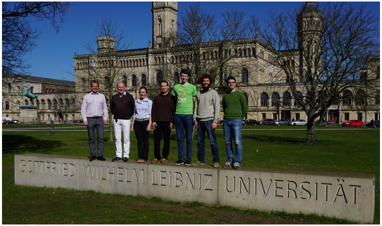
- Ideally, the BEC is in a single spatial mode.
- The state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.
- Our criterion must hadle this.

Violation of the criterion: entanglement is detected II



LHS/RHS for (top) our present work, and (bottom) for Science 2018.

Collaborators on entanglement conditions for double-well Dicke states



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Summary

- Detection of bipartite entanglement and EPR steering close to Dicke states. It works also for split spin-squeezed states.
- Non-symmetric states within the wells and a varying particle number can also be handled.

G. Vitagliano, I. Apellaniz, M. Fadel, M. Kleinmann,
B. Lücke, C. Klempt, and G. Tóth,
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