Witnessing metrologically useful multiparticle entanglement

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Why multipartite entanglement and metrology are important?

- Full tomography is not possible, we still have to say something meaningful.

- Claiming “entanglement” is not sufficient for many particles.

- We should tell
  - How entangled the state is
  - What the state is good for, etc.
1 Introduction and motivation

2 Spin squeezing and entanglement
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion

3 Detecting metrologically useful entanglement
   - Basics of quantum metrology
   - Witnessing metrological usefulness
   - Metrology with measuring $\langle J_z \rangle$
   - Metrology with measuring $\langle J_z^2 \rangle$
   - Metrology with measuring any operator
Entanglement

A state is (fully) separable if it can be written as

\[ \sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes \ldots \otimes \varrho_k^{(N)}. \]

If a state is not separable then it is entangled (Werner, 1989).
A pure state is **k-producible** if it can be written as

\[ |\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \ldots \]

where \(|\Phi_i\rangle\) are states of at most \(k\) qubits.

A mixed state is **k-producible**, if it is a mixture of **k**-producible pure states.

[ e.g., Gühne, GT, NJP 2005. ]

- If a state is not **k**-producible, then it is at least \((k + 1)\)-particle entangled.

![two-producible](image1)

![three-producible](image2)
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For spin-\(\frac{1}{2}\) particles, we can measure the collective angular momentum operators:

\[
J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma^{(k)}_l,
\]

where \(l = x, y, z\) and \(\sigma^{(k)}_l\) are Pauli spin matrices.

We can also measure the variances

\[
(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.
\]
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The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

\[ \xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}. \]

If \( \xi_s^2 < 1 \) then the state is entangled. [Sørensen, Duan, Cirac, Zoller, Nature (2001).]

- States detected are like this:
Let us assume that for a system we know only

\[ \vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle), \]
\[ \vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle). \]

A full set of generalized spin squeezing criteria is known for the case above.


Multipartite entanglement detection with spin squeezing (only two criteria!)

- Original spin-squeezing method

- Generalized method. BEC, 8000 particles. 28-particle entanglement is detected.

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Our main goals

- Detect *metrologically useful multipartite entanglement*, not just entanglement in general.

- Detect multipartite entanglement in the vicinity of *various* states.
Quantum metrology

- Fundamental task in metrology with a linear interferometer

We have to estimate $\theta$ in the dynamics

$$U = \exp(-iJ_l\theta)$$

where $l \in \{x, y, z\}$. 
Precision of parameter estimation

- Measure an operator $M$ to get the estimate $\theta$. The precision is

$$\langle \theta \rangle^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$
The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

\[(\Delta \theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \quad \frac{1}{(\Delta \theta)^2} \leq F_Q[\varrho, A].\]

where \(F_Q[\varrho, A]\) is the quantum Fisher information.

- The quantum Fisher information is given by an explicit formula for \(\varrho\) and \(A\).

\[F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k | A | l \rangle|^2,\]

where \(\varrho = \sum_k \lambda_k |k\rangle\langle k|\).
The quantum Fisher information vs. entanglement

- For separable states
  \[ F_Q[\rho, J_l] \leq N. \]
  [Pezze, Smerzi, PRL 2009; Hyllus, Gühne, Smerzi, PRA 2010]

- For states with at most \( k \)-particle entanglement (\( k \) is divisor of \( N \))
  \[ F_Q[\rho, J_l] \leq kN. \]
  [Hyllus et al., PRA 2012; GT, PRA 2012].

- If a state violates the above inequality then it has \((k + 1)\)-particle metrologically useful entanglement.
For separable states

\[(\Delta \theta)^2 \geq \frac{1}{N}.\]

[Pezze, Smerzi, PRL 2009; Hyllus, Gühne, Smerzi, PRA 2010]

For states with at most \(k\)-particle entanglement (\(k\) is divisor of \(N\))

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Witnessing metrological usefulness

Direct measurement of the sensitivity

- Measure \((\Delta \theta)^2\).
- Obtain bound on \(F_Q\) and multipartite entanglement, \(F_Q[\rho, A] \geq \frac{1}{(\Delta \theta)^2}\).
- Experimentally challenging, since we need quantum dynamics.
- The precision is affected by the noise during the dynamics.

[Experiments in cold atoms by the groups of M. Oberthaler, C. Klempt; photonic experiments of the Weinfurter group.]

Witnessing (our choice)

- Estimate how good the precision were, if we did the metrological process.
- Assume a perfect metrological process. Characterizes the state only.
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Pezze-Smerzi bound

\[(\Delta \theta)^2 = \frac{(\Delta J_z)^2}{|\partial_\theta \langle J_z \rangle|^2} = \frac{(\Delta J_z)^2}{\langle J_x \rangle^2} = \frac{\xi_s^2}{N}.\]

We measure \(\langle J_z \rangle\).

[Pezze, Smerzi, PRL 2009.]
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Metrology with Dicke states

- For Dicke state

\[ \langle J_l \rangle = 0, \; l = x, \; y, \; z, \; \langle J_z^2 \rangle = 0, \; \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}. \]

- Linear metrology

\[ U = \exp(-iJ_y \theta). \]

- Measure \( \langle J_z^2 \rangle \) to estimate \( \theta \). (We cannot measure first moments, since they are zero.)
Formula for maximal precision II

Maximal precision with a closed formula

\[
(\Delta \theta)^2_{\text{opt}} = \frac{2 \sqrt{(\Delta J_z^2)^2 (\Delta J_x^2)^2 + 4\langle J_y^2 \rangle - 3\langle J_y^2 \rangle - 2\langle J_y^2 \rangle (1+\langle J_x^2 \rangle) + 6\langle J_z J_x J_z \rangle}}{4(\langle J_x^2 \rangle - \langle J_z^2 \rangle)^2}.
\]

- Collective observables, like in the spin-squeezing criterion.
- Metrological usefulness can be verified without carrying out the metrological task.
- Tested on experimental data.

[ Apellaniz, Lücke, Peise, Klempt, GT, NJP 2015. ]
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Large step: we do not assume any metrological scheme

- We would like to know how good a state is for quantum metrology.
- We allow any operator to be measured for parameter estimation.
- Thus, we need to witness the quantum Fisher information.
We would like to measure the quantum Fisher information.

Related problem: For systems in thermal equilibrium

\[ F_Q(T) = \frac{4}{\pi} \int_0^\infty d\omega \tanh \left( \frac{\omega}{2T} \right) \chi''(\omega, T) \]

Needs measuring the imaginary part of the dynamic susceptibility, \( \chi'' \), as a function of \( \omega \).

Hauke et al., Nat. Phys. 12, 778 (2016).

We have systems not in thermal equilibrium, and can measure only a few operators.
The quantum Fisher information is the convex roof of the variance

\[ F_Q[\rho, A] = 4 \min_{\rho_k, \psi_k} \sum_k \rho_k (\Delta A)^2_k, \]

where

\[ \rho = \sum_k \rho_k |\psi_k\rangle \langle \psi_k|. \]

Thus, it is similar to entanglement measures that are also defined by convex roofs.
Legendre transform

- Optimal linear lower bound on a convex function $g(\rho)$ based on an operator expectation value $w = \langle W \rangle_\rho = \text{Tr}(W\rho)$

  $$g(\rho) \geq rw - \text{const.},$$

  where $w = \text{Tr}(\rho W)$.

- For every slope $r$ there is a "const."

- Textbooks say

  $$g(\rho) \geq B(w) := rw - \hat{g}(rW),$$

  where $\hat{g}$ is the Legendre transform

  $$\hat{g}(W) = \sup_\rho [\langle W \rangle_\rho - g(\rho)].$$

  [Gühne, Reimpell, Werner, PRL 2007; Eisert, Brandao, Audenaert, NJP 2007.]
Bound is best if we optimize over $r$ as

$$g(\rho) \geq \mathcal{B}(w) := \sup_r [rw - \hat{g}(rW)],$$

where again $w = \text{Tr}(\rho W)$.

$F_Q$ is the convex roof of the variance. Hence, it is sufficient to carry out an optimization over pure states

$$\hat{g}(W) = \sup_\psi [\langle W|\psi \rangle - g(\psi)].$$

Similar simplification has been used for entanglement measures.

[Gühne, Reimpell, Werner, PRL 2007; Eisert, Brandao, Audenaert, NJP 2007.]
For our case, the Legendre transform is

\[ \hat{F}_Q(W) = \sup_{\Psi} [\langle W - 4J_l^2 \rangle_\Psi + 4 \langle J_l \rangle^2_\Psi]. \]

With further simplifications, an optimization over a single real variable is needed

\[ \hat{F}_Q(W) = \sup_{\mu} \left\{ \lambda_{\text{max}} \left[ W - 4(J_l - \mu)^2 \right] \right\}. \]
The quantum Fisher information is the ideal quantity for using the Legendre transform technique.
Witnessing the quantum Fisher information based on the fidelity

Let us bound the quantum Fisher information based on some measurements. First, consider small systems. [See also Augusiak et al., 1506.08837.]

Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$. [Apellaniz et al., PRA 95, 032330 (2017).]
Bounding the qFi based on collective measurements

Bound for the quantum Fisher information for spin squeezed states (Pezze-Smerzi bound)

\[ F_Q[\rho, J_y] \geq \frac{\langle J_z \rangle^2}{(\Delta J_x)^2}. \]

[Pezze, Smerzi, PRL 2009.]
Bounding the qFi based on collective measurements II

- Optimal bound for the quantum Fisher information $F_Q[\rho, J_y]$ for spin squeezing for $N = 4$ particles

[Apellaniz, Kleinmann, Gühne, GT, PRA 95, 032330 (2017).]
Bounding the qFi based on collective measurements III

- Optimal bound for the quantum Fisher information $F_Q[\rho, J_y]$ for spin squeezing for $N = 4$ particles

On the bottom part of the figure $[(\Delta J_x)^2 < 1]$ the bound is very close to the Pezze-Smerzi bound!

[Apellaniz, Kleinman, Gühne, GT, PRA 95, 032330 (2017).]
The bound can be obtained if additional expectation value, i.e., $\langle J_x^2 \rangle$ is measured, or we assume symmetry:

[Apellaniz, Kleinman, Gühne, GT, PRA 95, 032330 (2017).]
Spin squeezing experiment

- Experiment with $N = 2300$ atoms,
  \[ \xi_s^2 = -8.2\text{dB} = 10^{-8.2/10} = 0.1514. \]


- The Pezze-Smerzi bound is:
  \[ \mathcal{F}_Q[\rho_N, J_y] \geq \frac{1}{N} \geq \frac{1}{\xi_s^2} = 6.605. \]

  We get the same value for our method!

  [Pezze, Smerzi, PRL 2009]

- Similar calculations for Dicke state experiments!

  [Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt, PRL 2014.]
Lower bound on the quantum Fisher information with the variance and the purity

\[(\Delta J_i)^2 - \frac{1}{4} F_Q[\rho, J_i] \leq \frac{N^2}{2} [1 - \text{Tr}(\rho^2)].\]

[ G. Tóth, arXiv:1701.07461. ]
Summary

- We discussed a very flexible method to detect multipartite entanglement and metrological usefulness.
- We can choose a set of operators and the method gives an optimal lower bound on $F_Q$.

Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015);
Apellaniz, Kleinmann, Gühne, GT, Phys. Rev. A 95, 032330 (2017),
Editors’ Suggestion.

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