Multicopy metrology with many-particle quantum states

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Introduction

- A quantum state is *useful* for metrology if it can outperform separable states in the precision of parameter estimation.
- Quantum entanglement is required for metrological usefulness [1].
- But there are highly entangled pure states that are not useful [2], while weakly entangled bound entangled states can be useful [3, 4].
- Can all entangled states be made useful with the idea of activation [5]?

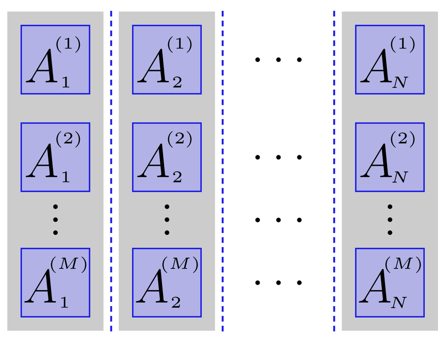

Figure 1: M copies of the N-partite state ϱ.

- Large class of entangled states become maximally useful in the limit of many copies.
- Non-useful states can be made useful by embedding into higher dimension.

Quantum Fisher information

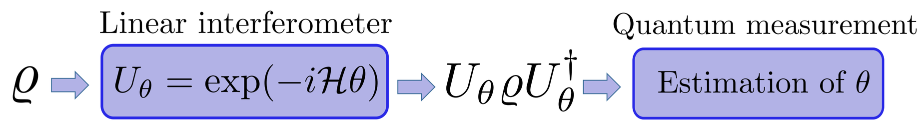

Figure 2: Typical process of quantum metrology

- $\mathcal{H}$ is assumed to be *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N, \qquad (1)$$

where h_n's are single-subsystem operators and $h_n = \otimes_{m=1}^{M} h_{A_n^{(m)}}$.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq 1/\mathcal{F}_Q[\varrho, \mathcal{H}], \qquad (2)$$

where the quantum Fisher information (QFI) is

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2, \qquad (3)$$

with $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ being the eigendecomposition. In general:

$$4(\Delta\mathcal{H})^2 \geq \mathcal{F}_Q[\varrho, \mathcal{H}] \geq 4I_\varrho(\mathcal{H}), \qquad (4)$$

with $I_\varrho(\mathcal{H}) = \mathrm{Tr}(\varrho\mathcal{H}^2) - \mathrm{Tr}(\sqrt{\varrho}\mathcal{H}\sqrt{\varrho}\mathcal{H})$.

Metrological gain

- We define the metrological gain compared to separable states, for a given Hamiltonian, by [6]

$$g_{\mathcal{H}}(\varrho) = \mathcal{F}_Q[\varrho, \mathcal{H}]/\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}), \qquad (5)$$

where the separable limit for *local* Hamiltonians is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2. \qquad (6)$$

- Eq. (5) can be maximized [6] over *local* Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\,\mathcal{H}} g_{\mathcal{H}}(\varrho). \qquad (7)$$

- Goal is to calculate the metrological gain $g_{\mathcal{H}}(\varrho^{\otimes M})$.
- Scaling propeties
  - Shot-noise scaling: for separable states $g_{\mathcal{H}} \sim 1$ ($\mathcal{F}_Q \sim N$) at best.
  - Heisenberg scaling: for entangled states $g_{\mathcal{H}} \sim N$ ($\mathcal{F}_Q \sim N^2$) at best.

Limit of many copies

Entangled states of $N \geq 2$ qudits of dimension d are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}. \qquad (8)$$

For the *proof*, use Eq. (4) and calculate $I_{\varrho^{\otimes M}}(\mathcal{H})$, where $h_n = (D^{\otimes M})_{A_n}$ with $D = \mathrm{diag}(+1, -1, +1, -1, \ldots)$ and

$$\varrho = \sum_{k,l=0}^{d-1} c_{kl} (|k\rangle\langle l|)^{\otimes N}. \qquad (9)$$

- *Example*:

$$\varrho_p = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2}, \qquad (10)$$

with $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$.

- *Example*: $c_{00} = c_{11} = 1/2$ and $d = 2$

$$I(c_{01}, N) = N^2[1 - (1 - 4|c_{01}|^2)^{M/2}]. \qquad (11)$$

- *Example*: All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k |k\rangle^{\otimes N}, \qquad (12)$$

with $\sum_k |\sigma_k|^2 = 1$.

Further examples

The state in Eq. (12) with $\sum_k |\sigma_k|^2 = 1$ is useful for $d \geq 3$ and $N \geq 3$.

- *Embedding into higher dimension*: The state

$$|\psi\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} \qquad (13)$$

is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [2]. But

$$\sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} + 0\,|2\rangle^{\otimes N} \qquad (14)$$

is always useful.

- *Example*: For $|\psi\rangle^{\otimes M}$ from Eq. (13) with $1/N = 4|\sigma_0\sigma_1|^2$:

$$\mathcal{F}_Q = 4N^2[1 - (1 - 1/N)^M]. \qquad (15)$$

- *Scaling*: $|\psi\rangle^{\otimes M}$ with $1/N = 4|\sigma_0\sigma_1|^2$:

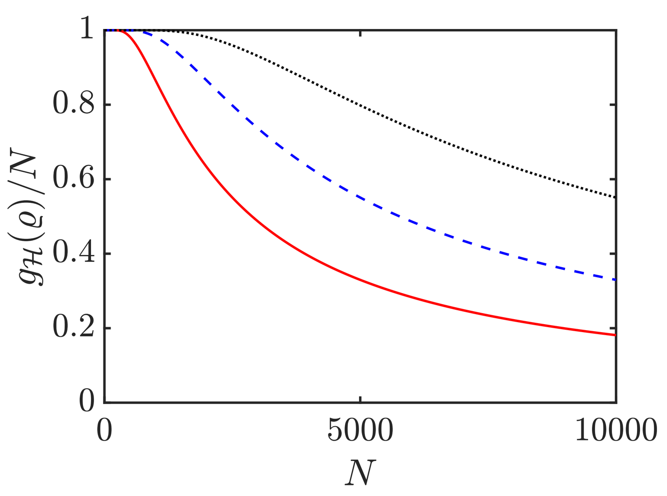

Figure 3: Dependence of the metrological gain on the particle number N for (solid) $M = 2000$, (dashed) 4000 and (dotted) 6000 copies.

White noise

Full-rank states of N qudits cannot be maximally useful in the infinite copy limit.

- *Example*: Isotropic state of two qubits

$$\varrho = p\,|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}| + (1-p)\mathbb{1}/2^2, \qquad (16)$$

where $|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

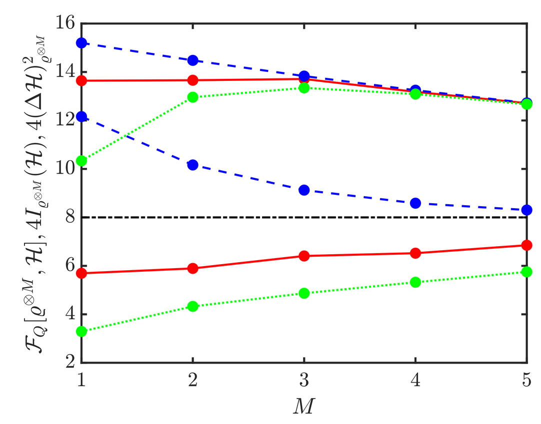

Figure 4: The QFI and the bounds from Eq. (4) as a function of M with $h_n = \sigma_z^{\otimes M}$. With $p = 0.9$ (top) and $p = 0.52$ (bottom).

- *Example*: Embedding the noisy GHZ

$$\varrho_p = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\mathbb{1}/2^N. \qquad (17)$$

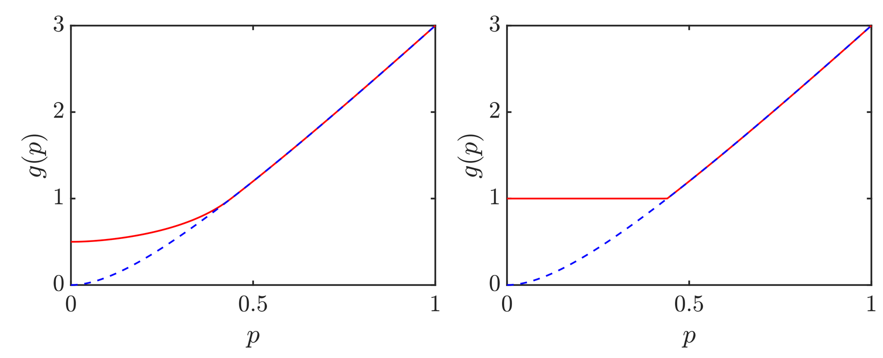

Figure 5: Embedding (solid) Eq. (17) with $N = 3$ into (left) $d = 3$, (right) $d = 4$.

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