Quantum Wasserstein distance based on an optimization over separable states

József Pitrik
HUN-REN Wigner Research Centre for Physics, Budapest
HUN-REN Alfréd Rényi Institute of Mathematics, Budapest
Department of Analysis and Operations Research, BUTE

Common work with Géza Tóth (UPV/EHU Bilbao)

QOT via quantum channels

The approach of De Palma and Trevisan[1]

- For any $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the set $\mathcal{M}(\rho, \sigma)$ of *quantum transport maps* from ρ to σ is the set of the quantum channels (CPTP maps) such that

$$\Phi : \mathcal{T}_1(\mathrm{supp}\,(\rho)) \to \mathcal{T}_1(\mathcal{H}), \quad \Phi(\rho) = \sigma.$$

- We can associate with any $\Phi \in \mathcal{M}(\rho, \sigma)$ the quantum state $\Pi_\Phi \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}^*)$ by

$$\Pi_\Phi = \left(\Phi \otimes I_{\mathcal{T}_1(\mathcal{H}^*)}\right)\left(||\sqrt{\rho}\rangle\rangle \, \langle\langle\sqrt{\rho}||\right).$$

[1] G. De Palma and D. Trevisan, *Quantum optimal transport with quantum channels*, Ann. Henri Poincaré **22** (2021), 3199–3234.

- Since
$$\mathrm{Tr}_{\mathcal{H}}\Pi_\Phi = \rho^T \quad \text{ad} \quad \mathrm{Tr}_{\mathcal{H}^*}\Pi_\Phi = \sigma,$$
where X^T is the transpose map, i.e. $X^T \langle\phi| = \langle\phi| X$, it induce the following definition:
- The set of **quantum couplings** assosiated with $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ is
$$\mathcal{C}(\rho, \sigma) = \{\Pi \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}^*) : \mathrm{Tr}_{\mathcal{H}}\Pi = \rho^T, \mathrm{Tr}_{\mathcal{H}^*}\Pi = \sigma\}.$$
- De Palma and Trevisan showed that for any $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the map $\Phi \mapsto \Pi_\Phi$ is a bijection between $\mathcal{M}(\rho, \sigma)$ and $\mathcal{C}(\rho, \sigma)$, that is in striking contrast to the classical case, the quantum couplings are in one-to-one correspondance with the quantum transport maps.
- Why? The primary reason: quantum channels **can "split mass"**, i.e. they can send pure states to mixed states.

- The **cost operator** for fixed self-adjoint operators $\{A_i\}_{i=1}^N$:

$$C = \sum_{j=1}^{N} \left(A_j \otimes I_{\mathcal{H}^*} - I_{\mathcal{H}} \otimes A_j^T \right)^2$$

- The transport cost for a coupling Π is

$$C(\Pi) = \operatorname{Tr}_{\mathcal{H}\otimes\mathcal{H}^*} \Pi C$$

- The **quantum Wasserstein (pseudo-)distance** $D_C(\rho, \sigma)$ is defined by

$$D_C^2(\rho, \sigma) = \inf_{\Pi \in \mathcal{C}(\rho,\sigma)} C(\Pi)$$

Some strange properties

- $D_C(\rho,\sigma) = D_C(\sigma,\rho)$ $\surd$
- If $\rho=\sigma$ then the optimal transport map corresponds to the identity map $\Phi = I$, so $D_C(\rho,\rho)^2 = C\left(\left|\left|\sqrt{\rho}\right\rangle\right\rangle\left\langle\left\langle\sqrt{\rho}\right|\right|\right)$ and

$$\begin{aligned} D_C(\rho,\rho)^2 &= -\sum_{i=1}^{N} \mathrm{Tr}\left([A_i,\sqrt{\rho}]^2\right) \\ &= 2\sum_{i=1}^{M}\left(\mathrm{Tr}\left(\rho A_i^2\right) - \mathrm{Tr}\left(\sqrt{\rho}A_i\sqrt{\rho}A_i\right)\right), \end{aligned}$$

  which is the famous **the Wigner – Yanase information**!
- For any $\rho,\tau,\sigma \in \mathcal{S}(\mathcal{H})$ the modified triangle inequality holds:

$$D_C(\rho,\sigma) \le D_C(\rho,\tau) + D_C(\tau,\tau) + D_C(\tau,\sigma).$$

Our contribution[2]

A bipartite quantum state is **separable** if it can be given as

$$\sum_k p_k |\Psi_k\rangle\langle\Psi_k| \otimes |\Phi_k\rangle\langle\Phi_k|,$$

with $\sum_k p_k = 1$. If a state cannot be written in this form, then it is called **entangled**. We denote the convex set of separable states by $\mathcal{S}_{sep}$. We define the modified **quantum Wasserstein (pseudo-)distance** by

$$D^2_{sep}(\rho, \sigma) = \inf_{\Pi} C(\Pi) = \inf_{\Pi} \sum_{j=1}^{N} \mathrm{Tr}\left(A_j \otimes I_{\mathcal{H}^*} - I_{\mathcal{H}} \otimes A_j^T\right)^2 \Pi,$$

where $\Pi \in \mathcal{C}(\rho, \sigma) \cap \mathcal{S}_{sep}$ are the separable couplings of the marginals ρ and σ.

[2] Géza Tóth, J.P. *Quantum Wasserstein distance based on an optimization over separable states,* Quantum 7 (2023), 1143

- For two qubits, **it is computable numerically** with semidefinite programming.
- In general,

$$D_{sep}(\rho, \sigma) \geq D(\rho, \sigma).$$

- If the relation

$$D_{sep}(\rho, \sigma) > D(\rho, \sigma)$$

holds, then all optimal Π for $D(\rho, \sigma)$ is entangled.

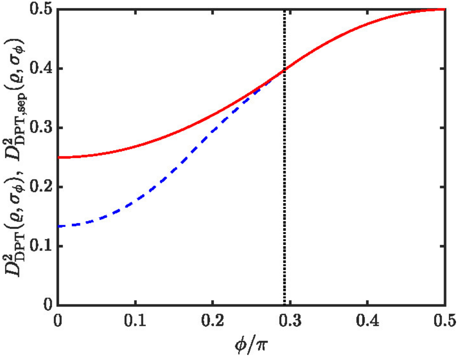
Let us consider the distance between two single-qubit mixed states

$$\rho = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{4}I,$$

and

$$\sigma_\phi = e^{-i\frac{\sigma_y}{2}\phi}\rho^{+i\frac{\sigma_y}{2}\phi},$$

for $N = 1$ and $A_1 = \sigma_z$.



Thus, an entangled Π can be cheaper than a separable one.

The modified self-distance

- For the self-distance in the modified case for $N = 1$ we get

$$D_{sep}(\rho, \rho)^2 = \frac{1}{4} F_Q[\rho, A],$$

where

$$F_Q[\rho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

the **quantum Fisher information** of the state $\rho = \sum_k \lambda_k |k\rangle\langle k|$ w.r.t the selfadjoint operator A.

- Note that

$$I_\rho(A) \leq \frac{1}{4} F_Q[\rho, A] \leq (\Delta A)^2_\rho,$$

where $I_\rho(A)$ is the Wigner-Yanase information and $(\Delta A)^2_\rho$ is the variance.

Petz's monotone metrics

- $\mathcal{D}_n^1$ invertible density matrices forms a smooth Riemannian manifold with tangent space at the footpoint ρ

$$\mathcal{T}_\rho \mathcal{D}_n^1 \equiv \{A \in M_n^{sa} : \operatorname{Tr} A = 0\}$$

- A Riemannian metric g_ρ with footpoint ρ on $\mathcal{D}_n^1$ is called **monotone metric** if

$$g_{T(\rho)}(T(A), T(A)) \leq g_\rho(A, A)$$

for all TPCP map T and $A \in \mathcal{T}_\rho \mathcal{D}_n^1$.

Petz's Theorem on characterisation of monotone metrics

- There exists a bijective correspondence between monoton metrics on $\mathcal{D}_n^1$ and operator monotone functions f on $(0, \infty)$, $f(1) = 1$ given by

$$g_\rho^f(A, B) = \langle A, (f(L_\rho R_\rho^{-1})R_\rho)^{-1} B \rangle_{HS},$$

where $L_\rho(A) = \rho A$, $R_\rho(A) = A\rho$, and $A, B \in \mathcal{T}_\rho \mathcal{D}_n^1$.

- A tipical element of $\mathcal{T}_\rho \mathcal{D}_n^1$ is $i[\rho, K]$, $(K \in M_n^{sa})$ and we can define the **metric adjusted skew information** by

$$I_\rho^f(K) = \frac{f(0)}{2} g_\rho^f(i[\rho, K], i[\rho, K]).$$

- With the choice $f(x) = \frac{(\sqrt{x}+1)^2}{4}$ we get the Wigner-Yanase information:
$$I_\rho^f(K) = -\mathrm{Tr}\left([K, \sqrt{\rho}]\right)^2.$$
- With the choice $f(x) = \frac{1+x}{2}$ we get the quantum Fisher information:
$$I_\rho^f(K) = F_Q[\rho, K] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|K|l\rangle|^2,$$
where $\rho = \sum_k \lambda_k |k\rangle\langle k|$.
- For a general f we can write explicitly:
$$I_\rho^f(K) = \frac{f(0)}{2} \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_l f(\lambda_k/\lambda_l)} |\langle k|K|l\rangle|^2.$$

Summary

- For the quantum Wasserstein distance, we restrict the optimization to separable states.
- Then, the self-distance is the quarter of the quantum Fisher information.
- We found a fundamental connection from quantum optimal transport to quantum entanglement theory and quantum metrology.

Thank you for your kind attention!