Multicopy metrology with many-particle quantum states

arXiv:2203.05538 (2022)

Róbert Trényi[1,2,3], Árpád Lukács[1,4,3], Paweł Horodecki[5,6], Ryszard Horodecki[5], Tamás Vértesi[7], and Géza Tóth[1,2,8,3]

[1]Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
[2]Donostia International Physics Center (DIPC), San Sebastián, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary
[4]Department of Mathematical Sciences, Durham University, Durham, United Kingdom
[5]International Centre for Theory of Quantum Technologies, University of Gdańsk, Gdańsk, Poland
[6]Faculty of Applied Physics and Mathematics, National Quantum Information Centre, Gdańsk University of Technology, Gdańsk, Poland
[7]Institute for Nuclear Research, Hungarian Academy of Sciences, Debrecen, Hungary
[8]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain

RDM 2022, June (Donostia-San Sebastián, Spain)

Table of Contents

Outline

1. Motivation
   - Harnessing entanglement
   - Quantum metrology
2. Improving metrological performance
   - Idea of activation
   - Embedding into higher dimension
   - Scaling properties

Entanglement

An N-partite quantum state is entangled if it cannot be written as

$$\varrho = \sum_i p_i \varrho_i^{(A_1)} \otimes \varrho_i^{(A_2)} \cdots \otimes \varrho_i^{(A_N)}.$$

Required for quantum advantage

- Quantum teleportation
- Superdense coding
- Quantum secure communication
- Quantum metrology

Outline

1. Motivation
   - Harnessing entanglement
   - Quantum metrology
2. Improving metrological performance
   - Idea of activation
   - Embedding into higher dimension
   - Scaling properties

Basic task in quantum metrology

Linear interferometer — Quantum measurement

$$\varrho \Rightarrow \boxed{U_\theta = \exp(-i\mathcal{H}\theta)} \Rightarrow U_\theta \varrho U_\theta^\dagger \Rightarrow \boxed{\text{Estimation of } \theta}$$

- $\mathcal{H}$ is *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N$$

where h_n's are single-subsystem operators.

Basic task in quantum metrology

Linear interferometer — Quantum measurement

$$\varrho \Rightarrow \boxed{U_\theta = \exp(-i\mathcal{H}\theta)} \Rightarrow U_\theta \varrho U_\theta^\dagger \Rightarrow \boxed{\text{Estimation of } \theta}$$

- $\mathcal{H}$ is *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N$$

  where h_n's are single-subsystem operators.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{\mathcal{F}_Q[\varrho, \mathcal{H}]},$$

  where the quantum Fisher information is

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2,$$

  with $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ being the eigendecomposition.

Metrological gain

- For a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

where the separable limit for *local* Hamiltonians is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2.$$

Metrological gain

- For a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

where the separable limit for *local* Hamiltonians is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2.$$

- $g_{\mathcal{H}}(\varrho)$ can be maximized over *local* Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho).$$

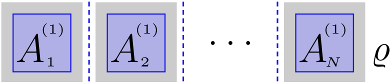
- If $g(\varrho) > 1$ then the state is useful metrologically.
  [G. Tóth et al., PRL 125, 020402 (2020)]

Motivation

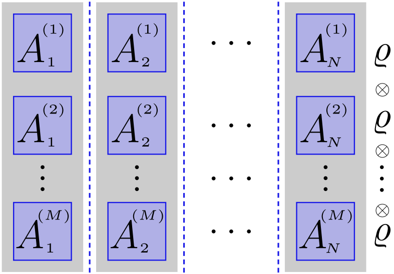
- Entanglement is required for usefulness
- Some highly entangled (pure) states are not useful
  [P. Hyllus et al., PRA 82, 012337 (2010)]
- But some weakly entangled states can be useful
  [G. Tóth and T. Vértesi, PRL 120, 020506 (2018)]
- What kind of states can be made useful with extended techniques?

Outline

Can considering more copies of a state ϱ help?

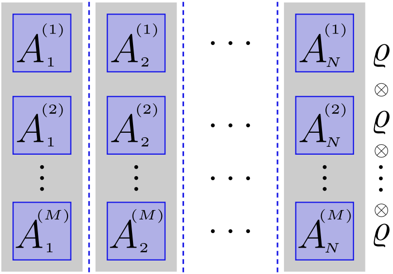


Can considering more copies of a state ϱ help?



The considered setting

Can considering more copies of a state ϱ help?



Can we have $g(\varrho^{\otimes M}) > 1 \geq g(\varrho)$?
[G. Tóth et al., PRL 125, 020402 (2020)]

A special subspace

Observation

Entangled states of $N \geq 2$ qudits of dimension d are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}.$$

A special subspace

Observation

Entangled states of $N \geq 2$ qudits of dimension d are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}.$$

- *Proof.*—Consider a state from this subspace

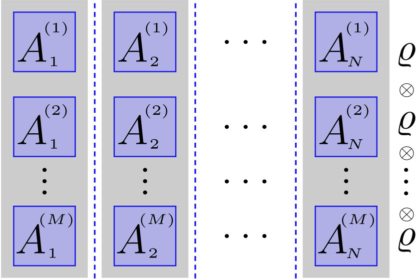
$$\varrho = \sum_{k,l=0}^{d-1} c_{kl} (|k\rangle\langle l|)^{\otimes N}$$

- Using the relation

$$\mathcal{F}_Q[\varrho, \mathcal{H}] \geq 4 I_\varrho(\mathcal{H}) = 4 \left[\mathrm{Tr}(\varrho \mathcal{H}^2) - \mathrm{Tr}(\sqrt{\varrho} \mathcal{H} \sqrt{\varrho} \mathcal{H}) \right].$$

- Computing $I_{\varrho^{\otimes M}}(\mathcal{H})$ with $\mathcal{H} = \sum_{n=1}^{N} (D^{\otimes M})_{A_n}$, where $D = \mathrm{diag}(+1, -1, +1, -1, ...)$.

A special subspace



$$
\begin{aligned}
h_1 &= D^{\otimes M} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}\\
h_2 &= \mathbb{1} \otimes D^{\otimes M} \otimes \cdots \otimes \mathbb{1}\\
&\vdots\\
h_n &= \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes D^{\otimes M}
\end{aligned}
$$

- $\mathcal{H} = \sum_{n=1}^{N} (D^{\otimes M})_{A_n}$, where $D = \mathrm{diag}(+1, -1, +1, -1, \ldots)$.

Examples

- The state

$$p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2},$$

with $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$.

Examples

- The state

$$p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2},$$

with $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$.

- For M copies of the state

$$\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2} + c_{01}(|0\rangle\langle 1|)^{\otimes N} + c_{01}^{*}(|1\rangle\langle 0|)^{\otimes N},$$

we have

$$4I(c_{01}, N) = 4N^2[1-(1-4|c_{01}|^2)^{M/2}].$$

Examples

- The state

$$p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2},$$

with $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$.

- For M copies of the state

$$\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2} + c_{01}(|0\rangle\langle 1|)^{\otimes N} + c_{01}^{*}(|1\rangle\langle 0|)^{\otimes N},$$

we have

$$4I(c_{01}, N) = 4N^2[1-(1-4|c_{01}|^2)^{M/2}].$$

- All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k\,|k\rangle^{\otimes N}.$$

White noise

Observation

Full-rank states of N qudits cannot be maximally useful in the infinite copy limit.

White noise

Observation

Full-rank states of N qudits cannot be maximally useful in the infinite copy limit.

- *Example*: Isotropic state of two qubits

$$\varrho^{(p)} = p\,|\Psi_{\rm me}\rangle\langle\Psi_{\rm me}| + (1-p)\mathbb{1}/2^2,$$

where $|\Psi_{\rm me}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

White noise

Observation

Full-rank states of N qudits cannot be maximally useful in the infinite copy limit.

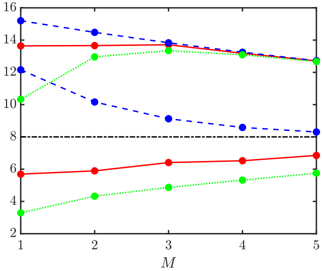
- *Example*: Isotropic state of two qubits

$$\varrho^{(p)} = p\,|\Psi_{\rm me}\rangle\langle\Psi_{\rm me}| + (1-p)\mathbb{1}/2^2,$$

where $|\Psi_{\rm me}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

- $\varrho^{(0.9)}$ (top 3 curves) and $\varrho^{(0.52)}$ (bottom 3 curves). $h_n = \sigma_z^{\otimes M}$.

$4(\Delta\mathcal{H})^2 \geq \mathcal{F}_Q[\varrho,\mathcal{H}] \geq 4I_\varrho(\mathcal{H})$



Outline

"GHZ"-like states

Observation

All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k \left|k\right\rangle^{\otimes N}$$

with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geq 3$ and $N \geq 3$.

"GHZ"-like states

Observation

All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k \left|k\right\rangle^{\otimes N}$$

with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geq 3$ and $N \geq 3$.

- The state for $N \geq 3$

$$|\psi\rangle = \sigma_0 \left|0\right\rangle^{\otimes N} + \sigma_1 \left|1\right\rangle^{\otimes N}$$

is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [P. Hyllus et al., PRA 82, 012337 (2010)].

“GHZ”-like states

Observation

All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k |k\rangle^{\otimes N}$$

with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geq 3$ and $N \geq 3$.

- The state for $N \geq 3$

$$|\psi\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N}$$

  is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [P. Hyllus et al., PRA 82, 012337 (2010)].
- But

$$|\psi'\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} + 0 |2\rangle^{\otimes N}$$

  is always useful.
- The non-useful $|\psi\rangle$, embedded into $d = 3$ ($|\psi'\rangle$) becomes useful.

Embedding mixed states

- Embedding the noisy GHZ state

$$\varrho_N^{(p)} = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{\mathbb{1}}{2^N}.$$

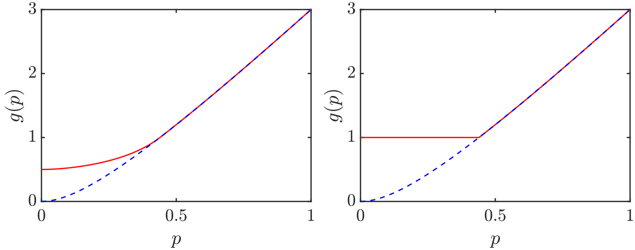


Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

Embedding mixed states

- Embedding the noisy GHZ state

$$\varrho_N^{(p)} = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{\mathbb{1}}{2^N}.$$

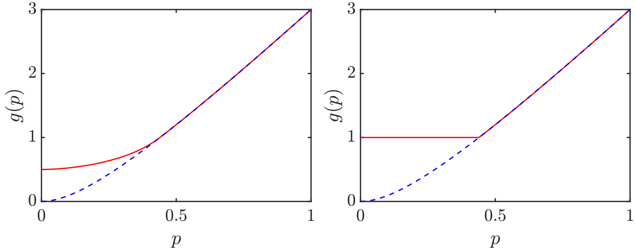


Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

- $\varrho_3^{(p)}$ is genuine multipartite entangled for $p > 0.428571$ [SM Hashemi Rafsanjani et al., PRA 86, 062303 (2012)].
- $\varrho_3^{(p)}$ is useful metrologically for $p > 0.439576$.

Outline

Scaling for a special pure state

- For separable states $g_{\mathcal{H}} \sim 1$ ($\mathcal{F}_Q \sim N$) at best (shot-noise scaling).
- For entangled states $g_{\mathcal{H}} \sim N$ ($\mathcal{F}_Q \sim N^2$) at best (Heisenberg scaling).

Scaling for a special pure state

- For separable states $g_{\mathcal{H}} \sim 1$ ($\mathcal{F}_Q \sim N$) at best (shot-noise scaling).
- For entangled states $g_{\mathcal{H}} \sim N$ ($\mathcal{F}_Q \sim N^2$) at best (Heisenberg scaling).

- $|\psi\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N}$ with $\frac{1}{N} = 4|\sigma_0\sigma_1|^2 \rightarrow \mathcal{F}_Q = 4N$ ($g = 1$).
- $|\psi\rangle^{\otimes M} \rightarrow \mathcal{F}_Q = 4N^2[1-(1-1/N)^M] \rightarrow g_{\mathcal{H}} = N[1-(1-1/N)^M]$

Scaling for a special pure state

- For separable states $g_{\mathcal{H}} \sim 1$ ($\mathcal{F}_Q \sim N$) at best (shot-noise scaling).
- For entangled states $g_{\mathcal{H}} \sim N$ ($\mathcal{F}_Q \sim N^2$) at best (Heisenberg scaling).

- $|\psi\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N}$ with $\frac{1}{N} = 4|\sigma_0\sigma_1|^2 \rightarrow \mathcal{F}_Q = 4N$ $(g = 1)$.
- $|\psi\rangle^{\otimes M} \rightarrow \mathcal{F}_Q = 4N^2[1-(1-1/N)^M] \rightarrow g_{\mathcal{H}} = N[1-(1-1/N)^M]$

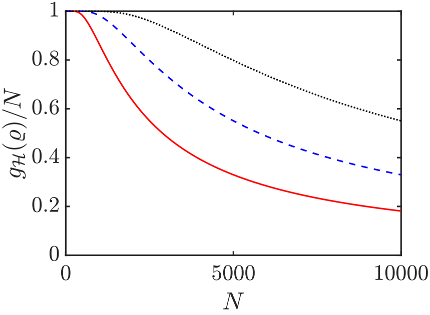


Figure: Dependence of the metrological gain on the particle number N for (solid) $M = 2000$, (dashed) 4000 and (dotted) 6000 copies. $h_n = \sigma_z^{\otimes M}$

Conclusions

- Investigated metrological performance of different quantum states when we have more copies of them.

- Identified a subspace in which all the states become useful if sufficiently many copies are taken.

- Also improved metrological performance by embedding.

See arXiv:2203.05538 (2022)!

Thank you for the attention!

QUANTERA

Alexander von Humboldt Stiftung/Foundation

MINISTERIO DE CIENCIA E INNOVACIÓN

EUSKO JAURLARITZA GOBIERNO VASCO

NATIONAL RESEARCH, DEVELOPMENT AND INNOVATION OFFICE