# Activation of metrologically useful genuine multipartite entanglement

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# Outline

# Different eras of quantum theory

- 1st quantum revolution
  - New laws governing nature
  - Explaining phenomena that was not possible before
    - Black-body radiation
    - Photoelectric effect
    - Understanding laser operation
    - Understanding semiconductors
- 2nd quantum revolution (nowadays)
  - Controlling quantum systems (cold atoms, trapped ions, Bose-Einstein condensates, photonic systems, ...)
  - Harnessing the properties of quantum theory
  - Quantum computing → speedups in different algorithms → Shor's, Grover's
  - Quantum communication → quantum key distribution → unconditional secrecy → BB84, E91, ... protocols
  - Quantum metrology → improving the precision of parameter estimation

Common point: entanglement is required for quantum advantage

# Outline

# Definition of (genuine multipartite) entanglement

## Definition

An $N$-partite quantum state is entangled if it cannot be written as

$$\varrho = \sum_i p_i \varrho_i^{(A_1)} \otimes \varrho_i^{(A_2)} \otimes \cdots \otimes \varrho_i^{(A_N)}.$$

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## Examples

- $N = 2$: $|00\rangle + |11\rangle$
- $N = 3$: $(|00\rangle + |11\rangle) \otimes |0\rangle$ and $|000\rangle + |111\rangle$

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## Examples

- $N = 4$: 3-entangled state $(|000\rangle + |111\rangle) \otimes (|0\rangle + |1\rangle)$
- $N = 5$: 3-entangled state $(|000\rangle + |111\rangle) \otimes (|00\rangle + |11\rangle)$
- $N = 6$: 2-entangled state $(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$

# Definition of (genuine multipartite) entanglement II.

## Definition

An $N$-partite *pure* quantum state $|\psi\rangle$ is biseparable if it can be written as

$$|\psi\rangle = |\psi_{\mathcal{A}}\rangle \otimes |\psi_{\mathcal{B}}\rangle \,,$$

where $\mathcal{A}, \mathcal{B}$ denote non-empty, complementary subsets of the $N$ parties.
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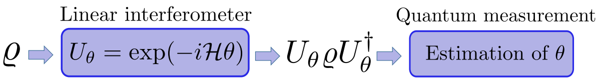
- $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|00...0\rangle + |11...1\rangle)$
- $|\mathrm{W}\rangle = \frac{1}{\sqrt{N}}(|00...1\rangle + ... + |01...0\rangle + |10...0\rangle)$

# Outline

# Basic task in quantum metrology




- $\mathcal{H}$ is *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N$$

where $h_n$'s are single-subsystem operators of the $N$-partite system.

# Basic task in quantum metrology

Linear interferometer → Quantum measurement

$$\varrho \Rightarrow \boxed{U_\theta = \exp(-i\mathcal{H}\theta)} \Rightarrow U_\theta \varrho U_\theta^\dagger \Rightarrow \boxed{\text{Estimation of } \theta}$$

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- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq \frac{1}{\mathcal{F}_Q[\varrho, \mathcal{H}]},$$

  where the quantum Fisher information is

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2,$$

  with $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ being the eigendecomposition.

# Some properties of the quantum Fisher information

- Lower and upper bounds

$$4(\Delta\mathcal{H})^2_\varrho \geq \mathcal{F}_Q[\varrho,\mathcal{H}] \geq 4I_\varrho(\mathcal{H}),$$

where $I_\varrho = \mathrm{Tr}(\varrho\mathcal{H}^2) - \mathrm{Tr}(\sqrt{\varrho}\mathcal{H}\sqrt{\varrho}\mathcal{H})$ is the Wigner-Yanase skew information.

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- General derivations yield: [G. Tóth and I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]
  - The maximum for separable states (shot-noise scaling)
    $\mathcal{F}_Q[\varrho,\mathcal{H}] \sim N \xRightarrow{\text{Cramér-Rao}} (\Delta\theta)^2 \sim 1/N$
  - The maximum for $k$-entangled states
    $\mathcal{F}_Q[\varrho,\mathcal{H}] \sim kN \xRightarrow{\text{Cramér-Rao}} (\Delta\theta)^2 \sim 1/kN$
  - The maximum for all quantum states (Heisenberg scaling)
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Quantum advantage means overcoming the shot-noise scaling.

# Experimental realizations

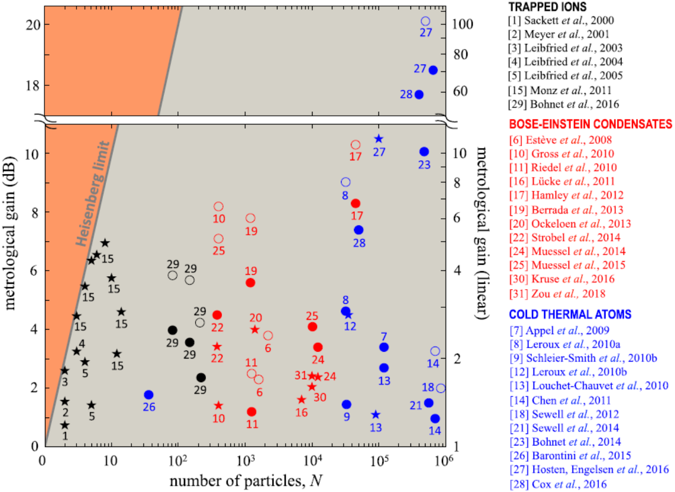
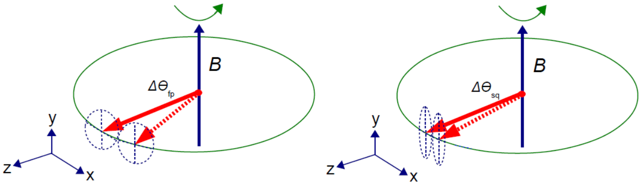



Figure from [L. Pezzè et al., Rev. Mod. Phys. 90, 035005 (2018)].

# Qualitative example with spin-squeezed states

- $N$ spin-1/2 particles in a magnetic field that points in the $y$ direction.
- Dynamics: $e^{-iJ_y\theta}$, where $\theta = \gamma Bt$ and $J_y = \frac{1}{2}\sum_{i=1}^{N}\sigma_y^{(i)}$.




fully polarized state (fp) spin-squeezed state (sq)

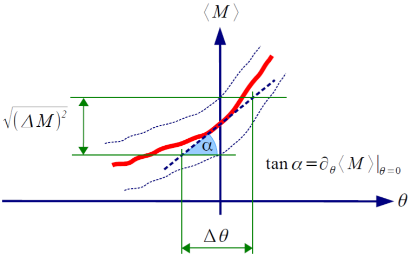
| state | $\vert +1/2\rangle^{\otimes N}$ | almost $\vert +1/2\rangle^{\otimes N}$ |
|---|---|---|
| $\langle J_z\rangle$ | $N/2$ | $\approx N/2$ |
| $(\Delta J_x)^2$ | $N/4$ | $< N/4$ |
| $(\Delta J_y)^2$ | $N/4$ | $> N/4$ |
| | fully separable | entangled |

Figure from [G. Tóth and I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)].

# Error propagation formula

- Measuring in the eigenbasis of $\mathcal{M}$ we get:

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{|\partial_\theta\langle\mathcal{M}\rangle|^2} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2}.$$

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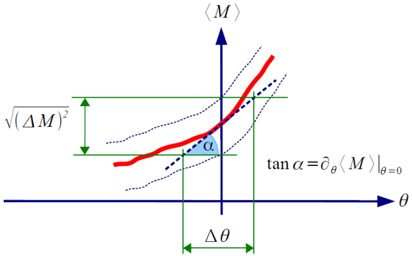


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- From the Cramér-Rao bound it follows that for any $\mathcal{M}$

$$\frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2} = (\Delta\theta)^2_{\mathcal{M}} \geq \frac{1}{\mathcal{F}_Q[\varrho,\mathcal{H}]}$$

# Outline

# Metrological gain

- For a given $\varrho$ and *local* Hamiltonian $\mathcal{H} = h_1 + \cdots + h_N$

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

← Performance of $\varrho$ with $\mathcal{H}$

← Best performance of all *separable* states with $\mathcal{H}$

where the separable limit is

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2.$$

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If $\sigma_{\mathrm{max/min}}(h_n) = \pm 1 \to \mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = 4N$ and the maximum of $\mathcal{F}_Q[\varrho, \mathcal{H}]$ is $4N^2$ for some entangled $\varrho$.

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- $g_{\mathcal{H}}(\varrho)$ can be maximized over *local* Hamiltonians

[G. Tóth et al., PRL 125, 020402 (2020)]

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho).$$

- If $g(\varrho) > 1$ then the state is useful metrologically.

# Relation to multipartite entanglement

- Fully-separable states $\rightarrow g \leq 1$ (shot-noise scaling).
- Entanglement is required for usefulness.
- Even weakly entangled states can be useful

  [G. Tóth and T. Vértesi, PRL 120, 020506 (2018)]
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- $g > k \rightarrow$ *metrologically useful* $(k+1)$-partite entanglement.
- $g > N-1 \rightarrow$ *metrologically useful* $N$-partite/genuine multipartite entanglement (GME).
- $g = N$ ($\mathcal{F}_Q = 4N^2$) is the maximal usefulness (Heisenberg scaling).

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- $g = N$ $(\mathcal{F}_Q = 4N^2)$ is the maximal usefulness (Heisenberg scaling).
- There are non-useful GME states [P. Hyllus et al., PRA 82, 012337 (2010)]
- What kind of entangled states can be made useful with extended techniques?

# Activation in quantum information

A state $\varrho$ does not have a certain property but $\varrho^{\otimes k}$ does for some $k > 1$. This is called activation.

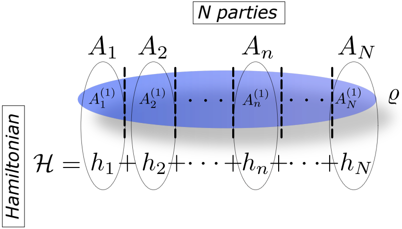
- Activating non-locality [M. Navascués and T. Vértesi, PRL 106, 060403 (2011)]
- Activating GME [H. Yamasaki et al., Quantum 6, 695 (2022)]
- Activating metrological usefulness [G. Tóth et al., PRL 125, 020402 (2020)]

# Outline

1. Introduction
   - Different eras of quantum theory
   - Multipartite entanglement
2. Quantum metrology
   - Main goal and quantum advantage
   - Characterizing metrological performance
3. Improving metrological performance
   - Taking many copies
   - Embedding into higher dimension

# Multi-copy scheme with interaction between copies

The single-subsystem operators $h_n$'s act between the copies:



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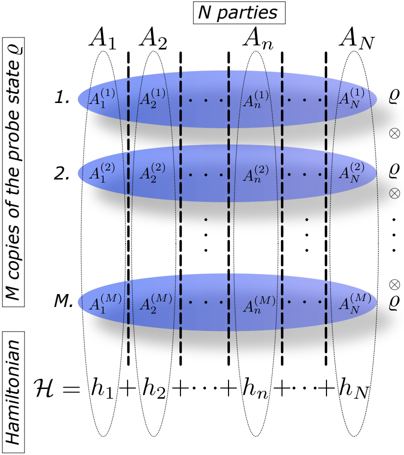

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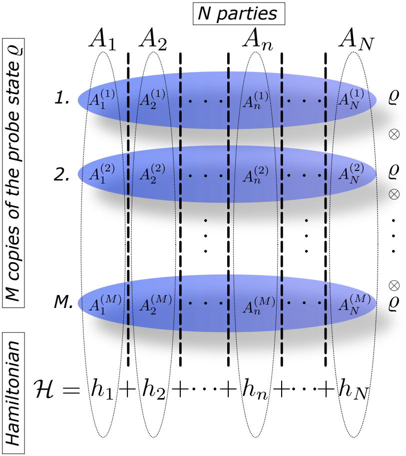


The gain can be improved $g(\varrho^{\otimes M}) > g(\varrho)$! [G. Tóth et al., PRL 125, 020402 (2020)]

# Metrologically useful GME activation

## Result

Entangled states of $N \geq 2$ qudits of dimension $d$ are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}.$$

The maximum is attained exponentially fast with the number of copies.

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$$\varrho = \sum_{k,l=0}^{d-1} c_{kl} (|k\rangle\langle l|)^{\otimes N}$$

$$h_n = D^{\otimes M}, \text{ for } 1 \leq n \leq N$$

$$D = \mathrm{diag}(+1, -1, +1, -1, ...)$$

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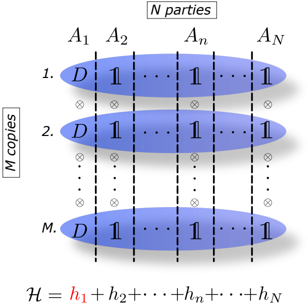

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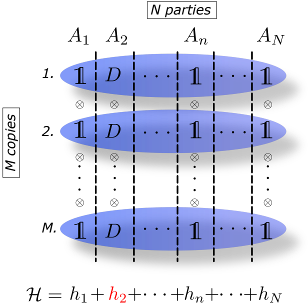

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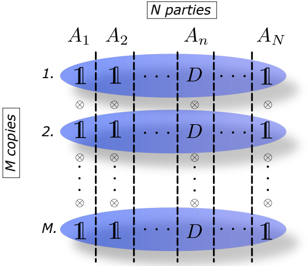


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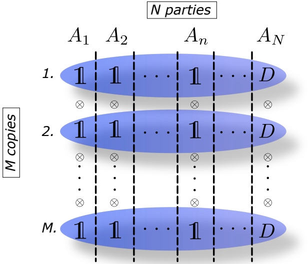


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# Examples

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$$\varrho_N(p) = p\,|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2}.$$

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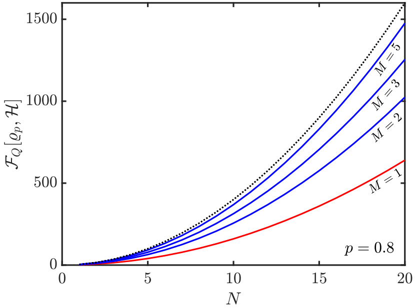

# An example for $N = 3$

Consider the state

$$\varrho_3(p) = p\,|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3| + \frac{1-p}{2}\left(|000\rangle\langle000| + |111\rangle\langle111|\right),$$

with $p = 0.8$.

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Consider the state

$$\varrho_3(p) = p\,|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3| + \frac{1-p}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111|\right),$$

with $p = 0.8$.

- 1-copy:

$$\mathcal{F}_Q[\varrho_3(p), \mathcal{H}_{M=1}] = 23.0400,$$

where $\mathcal{H}_{M=1} = \sigma_z^{(1)} + \sigma_z^{(2)} + \sigma_z^{(3)}$.

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- 2 copies:

  $$\mathcal{F}_Q[\varrho_3(p)^{\otimes 2}, \mathcal{H}_{M=2}] = 28.0976,$$

  where $\mathcal{H}_{M=2} = \sigma_z^{(1)}\sigma_z^{(4)} + \sigma_z^{(2)}\sigma_z^{(5)} + \sigma_z^{(3)}\sigma_z^{(6)}$.

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}_{M=1}) = \mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}_{M=2}) = 12.$$

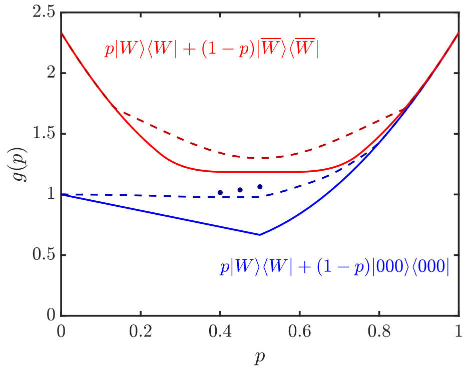
# States outside the previous subspace

- For $N = 3$ with the states

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

$$|\overline{W}\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle)$$

- Using the numerical optimization for $g(\varrho)$ [G. Tóth et al., PRL 125, 020402 (2020)].



# Phase noise for $N = 3$, $M = 1$ copy

$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ with $\mathcal{H} = h_1 + h_2 + h_3$, where $h_n = \sigma_z^{\otimes M}$.

**For $M = 1$ copy:**

$$\begin{aligned} \mathcal{F}_Q[|\mathrm{GHZ}\rangle, \mathcal{H}] &= 36 = 4N^2 \,(\mathrm{maximal}), \\ \mathcal{F}_Q[\varrho, \mathcal{H}] &< 36, \end{aligned}$$

with

$$\varrho = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\,|\mathrm{GHZ}_\phi\rangle\langle\mathrm{GHZ}_\phi|\,,$$

where $|\mathrm{GHZ}_\phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + e^{-i\phi}\,|111\rangle)$.

- So $\varrho$ is a mixture of $|\mathrm{GHZ}\rangle$ and the phase-error affected $|\mathrm{GHZ}\rangle$.
- For 1 copy, the quantum Fisher information decreases if there is a phase-error.

# Tolerating phase noise for $N = 3$, $M = 3$ copies

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**For $M = 3$ copies:**

$$\begin{aligned} \mathcal{F}_Q[|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \mathcal{H}] &= 36 = 4N^2 \,(\mathrm{maximal}), \\ \mathcal{F}_Q[\varrho, \mathcal{H}] &= 36, \end{aligned}$$

where $\varrho$ is some mixture of states with phase-error on at most 1 copy:

$$\begin{aligned} &|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \\ &|\mathrm{GHZ}_{\phi_1}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \\ &|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_2}\rangle \otimes |\mathrm{GHZ}\rangle, \\ &|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_3}\rangle. \end{aligned}$$

- For 3 copies, the quantum Fisher information stays maximal if there is a phase-error on at most 1 copy.
- Adding more copies protects against phase-error on 1 copy.

# Outline

# “GHZ”-like states

## Result

All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k \left|k\right\rangle^{\otimes N}$$

with $\sum_k |\sigma_k|^2 = 1$ are useful for $d \geq 3$ and $N \geq 3$.

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- The state for $N \geq 3$ with $d = 2$

$$|\psi\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N}$$

is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [P. Hyllus et al., PRA 82, 012337 (2010)].

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- But with $d = 3$

$$|\psi'\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} + 0\,|2\rangle^{\otimes N}$$

is always useful.

- The non-useful $|\psi\rangle$, embedded into $d = 3$ ($|\psi'\rangle$) becomes useful.

# Conclusions

- Investigated the metrological performance of quantum states in the multicopy scenario.
- Identified a subspace in which metrologically useful GME activation is possible.
- Also improved metrological performance by embedding.

See New J. Phys. 26 023034 (2024)!

Thank you for the attention!

QUANTERA

Alexander von Humboldt Stiftung/Foundation

MINISTERIO DE CIENCIA E INNOVACIÓN

EUSKO JAURLARITZA GOBIERNO VASCO

NATIONAL RESEARCH, DEVELOPMENT AND INNOVATION OFFICE

# Optimal measurements

- In the limit of many copies ($M \gg 1$)

$$\mathcal{F}_Q[\varrho_N(p)^{\otimes M}, \mathcal{H}] = 4N^2 \implies (\Delta\theta)^2 \geq 1/\mathcal{F}_Q[\varrho_N(p)^{\otimes M}, \mathcal{H}] = 1/4N^2$$

- Can we actually reach this limit with simple measurements?
- Measuring in the eigenbasis of $\mathcal{M}$ (error propagation formula):

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{|\partial_\theta \langle \mathcal{M} \rangle|^2} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M}, \mathcal{H}] \rangle^2}.$$

- For $M$ copies of $\varrho_N(p)$ we constructed a simple $\mathcal{M}$ such that

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{1 + (M-1)p^2}{4MN^2p^2}$$

- For $M = 2$ copies of $\varrho_3(p)$

$$\mathcal{M} = \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} + \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$$

# The general measurements for Observation 1

$$\varrho(p,q,r) = p\,|\mathrm{GHZ}_q\rangle\langle\mathrm{GHZ}_q| + (1-p)[r(|0\rangle\langle 0|)^{\otimes N} + (1-r)(|1\rangle\langle 1|)^{\otimes N}],$$

with

$$|\mathrm{GHZ}_q\rangle = \sqrt{q}\,|000..00\rangle + \sqrt{1-q}\,|111..11\rangle\,,$$

The following operator, being the sum of $M$ correlation terms

$$\mathcal{M} = \sum_{m=1}^{M} Z^{\otimes(m-1)} \otimes Y \otimes Z^{\otimes(M-m)},$$

where we define the operators acting on a single copy

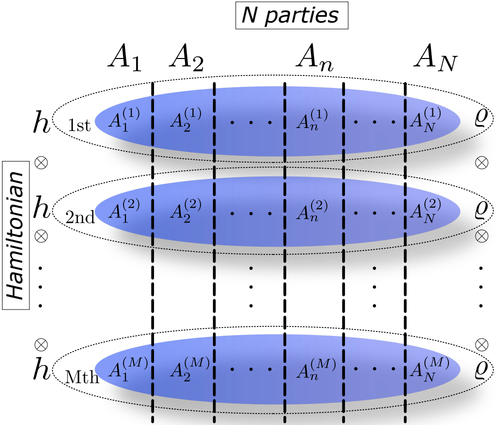
$$Y = \begin{cases} \sigma_y^{\otimes N} & \text{for odd } N, \\ \sigma_x \otimes \sigma_y^{\otimes(N-1)} & \text{for even } N, \end{cases}$$

$$Z = \sigma_z \otimes \mathbb{1}^{\otimes(N-1)}.$$

$$(\Delta\theta)^2_{\mathcal{M}} = \frac{1/[4q(1-q)] + (M-1)p^2}{4MN^2p^2}.$$

# Scheme without interaction between copies

Consider $M$ copies of an $N$-partite state $\varrho$, all undergoing a dynamics governed by the same Hamiltonian $h$:




$$\mathcal{F}_Q[\varrho^{\otimes M}, h^{\otimes M}] = M\mathcal{F}_Q[\varrho, h],$$

but the maximum for separable states also increases

$$\mathcal{F}_Q^{(\mathrm{sep})}(h^{\otimes M}) = M\mathcal{F}_Q^{(\mathrm{sep})}(h).$$

So the gain remains the same

$$g_{h^{\otimes M}}(\varrho^{\otimes M}) = g_h(\varrho).$$

No improvement in the gain!

## Observation

Full-rank states of $N$ qudits cannot be maximally useful in the infinite copy limit.

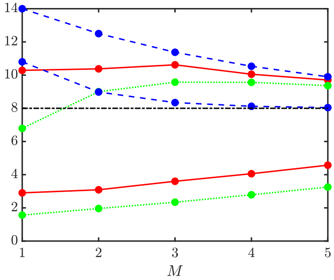
- *Example*: Isotropic state of two qubits

$$\varrho^{(p)} = p\,|\Psi_{\rm me}\rangle\langle\Psi_{\rm me}| + (1-p)\mathbb{1}/2^2,$$

where $|\Psi_{\rm me}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

- $\varrho^{(0.75)}$ (top 3 curves) and $\varrho^{(0.35)}$ (bottom 3 curves). $h_n = \sigma_z^{\otimes M}$.

$$4(\Delta\mathcal{H})^2 \geq \mathcal{F}_Q[\varrho,\mathcal{H}] \geq 4I_\varrho(\mathcal{H})$$



# Embedding mixed states

- Embedding the noisy GHZ state

$$\varrho_N^{(p)} = p\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)\frac{\mathbb{1}}{2^N}.$$

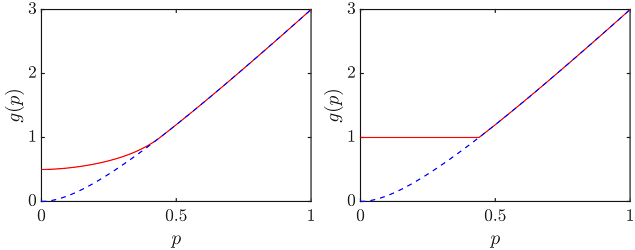



Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

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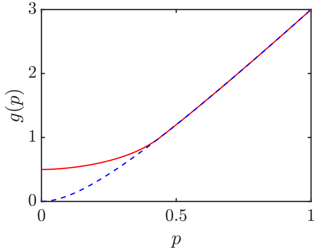


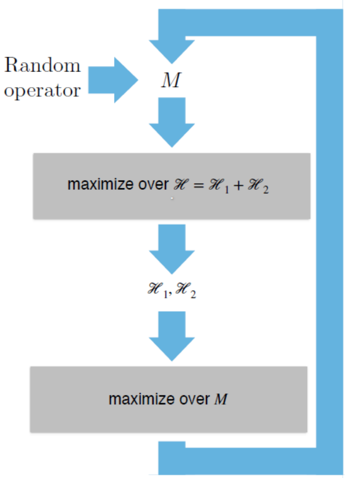

Figure: The metrological gain for the state $\varrho_3^{(p)}$ (dashed), embedded into $d = 3$ (left), $d = 4$ (right).

- $\varrho_3^{(p)}$ is genuine multipartite entangled for $p > 0.428571$ [SM Hashemi Rafsanjani et al., PRA 86, 062303 (2012)].
- $\varrho_3^{(p)}$ is useful metrologically for $p > 0.439576$.

# See-saw method for optimizing the gain

- Used in [G. Tóth et al., PRL 125, 020402 (2020)].
- Minimizing $(\Delta\theta)^2_{\mathcal{M}} = \frac{(\Delta\mathcal{M})^2}{\langle i[\mathcal{M},\mathcal{H}]\rangle^2} \geq \frac{1}{\mathcal{F}_Q[\varrho,\mathcal{H}]}$ with constraints $c_n\mathbf{1} \pm h_n \geq 0$.
- For given $\varrho$ and $\mathcal{H} = h_1 + h_2$ the symmetric logarithmic derivate gives the optimum

$$\mathcal{M}_{opt} = 2i\sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \ \langle k|\mathcal{H}|l\rangle$$



# A mixed biseparable state

$$\varrho = \frac{1}{3}\Big( |\phi^+\rangle\langle\phi^+|_{AB} \otimes |0\rangle\langle 0|_C + |\phi^+\rangle\langle\phi^+|_{AC} \otimes |0\rangle\langle 0|_B + |\phi^+\rangle\langle\phi^+|_{BC} \otimes |0\rangle\langle 0|_A \Big),$$

where $|\phi^+\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$.

- Biseparable, thus not GME.
- Entangled across any cut.
- Can be prepared with $A, B$ preparing entanglement and then forgetting who had the entangled qubits.