Positivity violations of the density operator in the Hu-Paz-Zhang master equation

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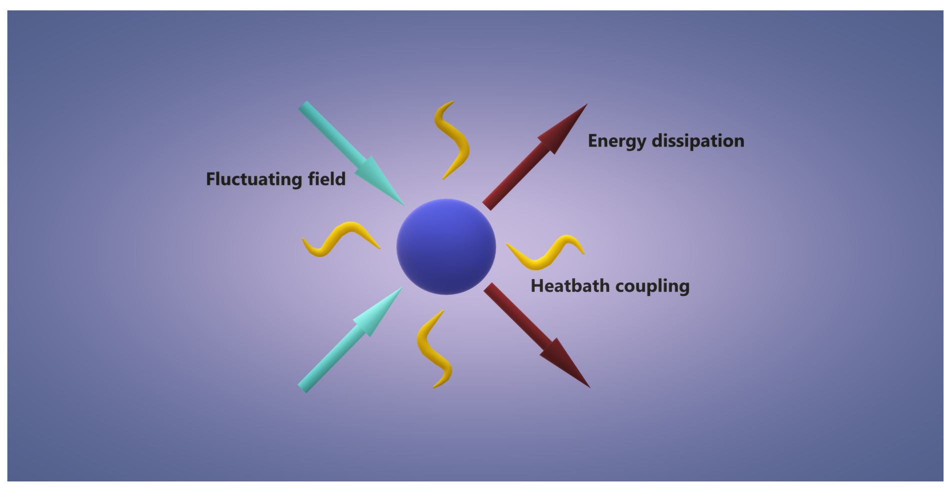
Introduction

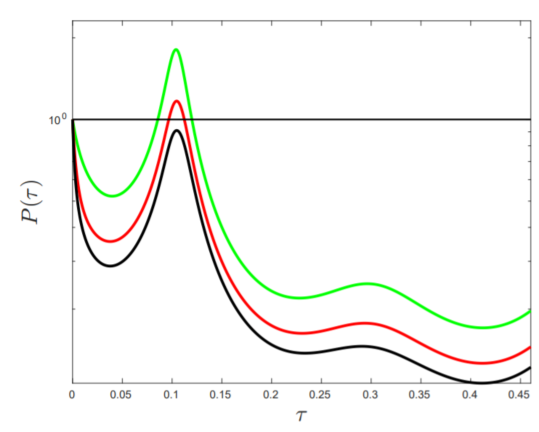
- Controlling open quantum systems is a difficult problem.
- A real quantum system $\mathcal{S}$ is not isolated.
- $\mathcal{S}$ interacts with the environment $\mathcal{R}$.
- Time evolution of the whole system $\mathcal{S}+\mathcal{R}$ is unitary with reversible dynamics.
- Time evolution of open system $\mathcal{S}$ alone is not unitary with irreversible dynamics.



- Quantum noise $\rightarrow$ decoherence $\Longleftrightarrow$ quantum information loss.

The model

- The main parameters are the temperature T and spectral density of the oscillator bath. We choose the ohmic spectral density with a Lorentz-Drude cutoff function. Parameters: coupling γ, cut-off Ω_c.
- Master equations are from [1, 2].
- We follow the evolution of the density operator $\hat{\rho}(t)$ by Eq. (1).
- We examined some master equations with and without Lindblad form [3, 4, 5, 6, 7].
- Problem: If the master equation is derived only approximately, positivity of the density operator is not always guaranteed.



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Gaussian density operator

- The Hu-Paz-Zhang master equation for the central quantum harmonic oscillator:

$$i\hbar\frac{\partial\hat{\rho}}{\partial t}=\left[\frac{\hat{p}^2}{2m}+\frac{m\omega_p^2(t)\hat{x}^2}{2},\hat{\rho}\right]-iD_{pp}(t)[\hat{x},[\hat{x},\hat{\rho}]]+\lambda(t)[\hat{x},\{\hat{p},\hat{\rho}\}]+2iD_{px}(t)[\hat{x},[\hat{p},\hat{\rho}]]. \quad (1)$$

- After a long time (in the Markovian limit) the coefficients $\omega_p^2(t),\lambda(t),D_{pp}(t)$ and $D_{px}(t)$ of the master equation become time independent. Non-Markovian coefficients are taken from [1].
- A Gaussian self-adjoint density matrix in the position representation:

$$\rho(x,y,t)=N\exp\Big(-A\,(x-y)^2-iB\,(x^2-y^2)-C\,(x+y)^2-iD(x-y)-E(x+y)\Big). \quad (2)$$

- The Gaussian parameters A, B, C, D, E and N are real, and time dependent.
- If we rewrite equation (1) in position representation the Gaussian form of (2) is preserved.
- Solution of (1) for $\hat{\rho}$ is a physical operator (positive semidefinite) if and only if

$$A\geq C>0 \quad (3)$$

We check its positivity by investigating the ratio A/C.

Our main results

- The non-Markovian equation is physical if the stationary solution is physical.

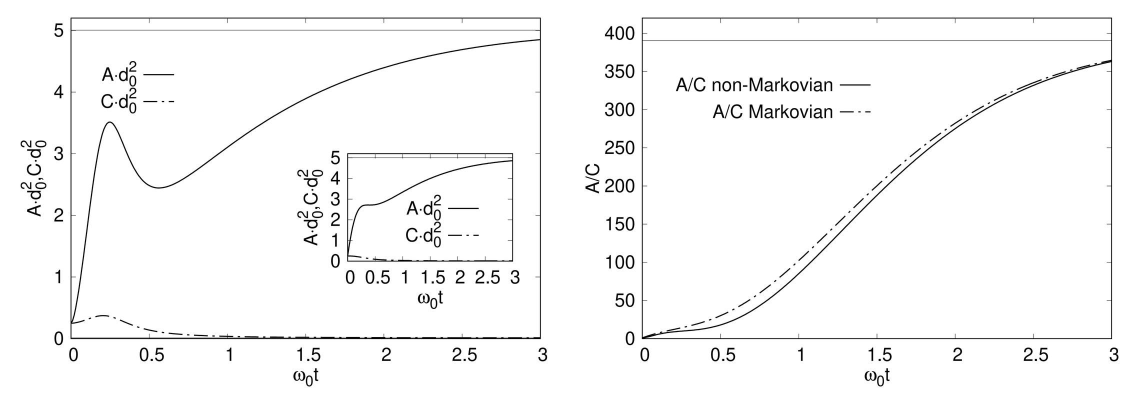

Figure 1

- Figure 1: no positivity violation (the inset figure shows the Markovian run).
- Figure 2, left panel: The stationary solutions are: region I (not physical), II (physical), III (full time evolution is physical).

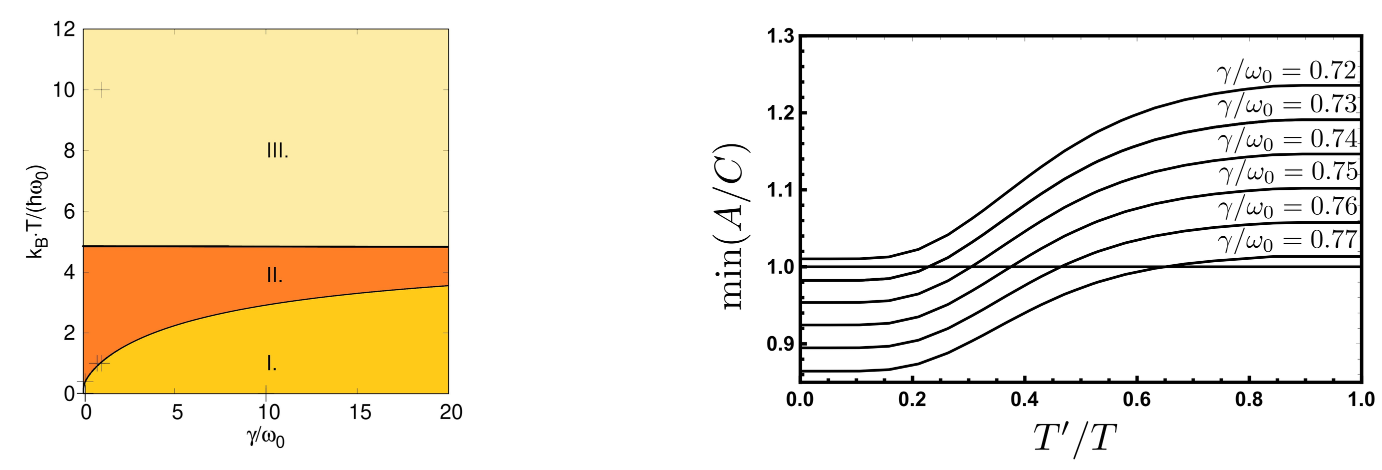

Figure 2

- Figure 2, right panel: Full time analysis of positivity in Markovian runs, if the time evolution of the central oscillator starts from a thermal state with temperature T' and the bath oscillators are also in thermal state with T.

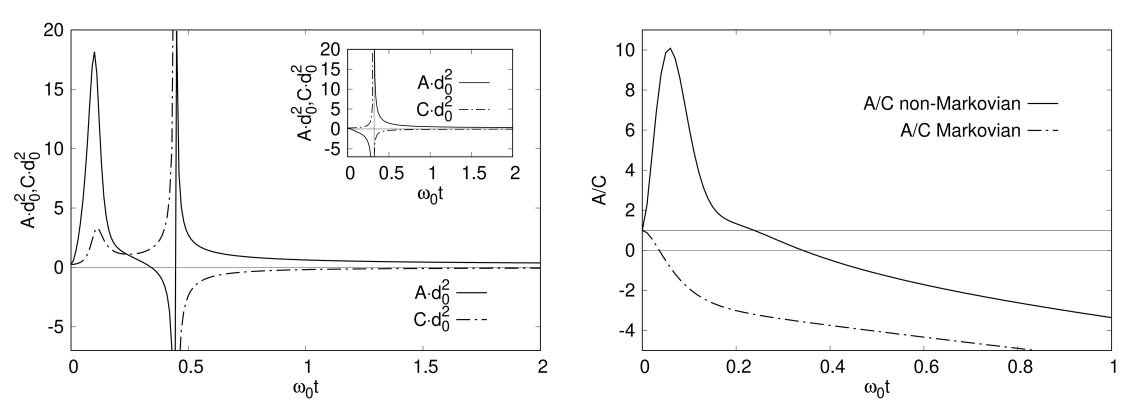

Figure 3

- Figure 3: the parameters belong to region I. The ratio of A/C goes below 1 (indicating positivity violation) and, at a later time, A changes sign and at an even further time, A and, C diverge, changing signs anew $\Longrightarrow$ $\mathrm{Tr}\hat{\rho}$ do not exists.