# Entanglement witnesses for detecting multi-qubit entanglement of many qubits (Motivations for quant-ph/0405165)

Quantum entanglement has been in the center of attention for quantum physicist for more than half a century. Bell inequalities [1–3] provide a very insightful approach for detecting an important characteristic arising from entanglement: non-locality. If a Bell inequality is violated for an experiment, then it means that the measurement results cannot be explained with a local hidden variable theory.

Bell inequalities have two uses: (i) they demonstrate that quantum mechanics as a theory is nonlocal. (ii) They prove that the quantum state created in the experiment is entangled. Every state violating a Bell inequality for some choice of observables is entangled. However, not all entangled states violate a Bell inequality [4]. Thus there are entangled states which allow for a local hidden variable model.

In the recent years powerful tools for detecting quantum entanglement were developed: *entanglement witnesses* [5]. These, in principle, make possible detecting any entangled states. Unlike Bell inequalities (which are classical), they use quantum mechanics for obtaining conditions for entanglement. This is the reason why in many situations *much fewer* measurements are sufficient for entanglement detection than with Bell inequalities.

Let us provide a simple example. The well-known CHSH inequality detects entangled states close to the state $\Psi = (|00\rangle + |11\rangle)/\sqrt{2}$. It requires measuring the spin components $x$ and $y$ for both qubits. The CHSH inequality claims that for states with local hidden variable model

$$\langle x_1 x_2 \rangle + \langle y_1 y_2 \rangle + \langle x_1 y_2 \rangle - \langle y_1 x_2 \rangle \leq 2, \tag{1}$$

where $\langle ... \rangle$ denotes expectation value. The maximum for local hidden variable models can be obtained by trying all the 16 possible combinations of $x_1, y_1, x_2, y_2 = \pm 1$. The quantum maximum of Eq. (1) is $2\sqrt{2}$. Thus in an experiment the visibility must be at least $2/(2\sqrt{2}) \approx 70\%$

How can one see that Bell inequalities do not use quantum mechanics? For example, they consider $\langle x_k \rangle = \langle y_k \rangle = +1$ a possible measurement outcome. But from quantum mechanics we know that $\langle x_k \rangle^2 + \langle y_k \rangle^2 \leq 1$. That is, the maximal length of a Bloch vector is one. If we use this knowledge then a much simpler condition can be obtained. So for separable (not entangled) states

$$\langle x_1 x_2 \rangle + \langle y_1 y_2 \rangle \leq 1. \tag{2}$$

This can be proved using that for product sates $\langle x_1 x_2 \rangle + \langle y_1 y_2 \rangle = \langle x_1 \rangle \langle x_2 \rangle + \langle y_1 \rangle \langle y_2 \rangle$. The condition given in Eq. (2) is basically an entanglement witness. If it is violated then the state is entangled. Here the quantum maximum is 2. Thus the robustness of this condition is larger: it requires only 50% visibility in an experiment. Other advantage: only two measurement settings are needed instead of four.

In the multi-qubit case the situation is the following [3]. Let us consider GHZ states for simplicity [6]. The Mermin inequality [2] can be used for entanglement detection around GHZ states. It also requires detecting $x$ and $y$ for each qubit. The condition for local hidden variable models for $n$ qubits is given as

$$\begin{aligned}
& \langle x_1 x_2 x_3 x_4 \cdots x_{N-1} x_N \rangle \\
- & \langle y_1 y_2 x_3 x_4 \cdots x_{N-1} x_N \rangle \\
+ & \langle y_1 y_2 y_3 y_4 \cdots x_{N-1} x_N \rangle \\
- & ... \\
+ & \langle y_1 y_2 y_3 y_4 \cdots y_{N-1} y_N \rangle \\
\leq & \; 2^{\lfloor n/2 \rfloor},
\end{aligned} \tag{3}$$

where $\lfloor x \rfloor$ denotes integer part. Here each term represents the sum of all its possible permutations. Very many measurements ... What actually matters, is not the number of terms, but the number of measurement settings. Here each term needs a separate setting. That is, there are no two terms which could be measured with the same setting.

After the long introduction we reached the main point. It is possible to design much more efficient conditions for entanglement using entanglement witnesses. These detect multi-qubit entanglement with much much fewer settings. In fact, they need only two settings. This is a surprise. It is possible since the states most often considered in quantum information (GHZ states, cluster states) are so-called *stabilizer states*. Thus stabilizer theory, already very much used in quantum error correction, can also be used for detecting entanglement. For details please see Ref. [7].


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