Supplemental Material for "Activation of metrologically useful genuine multipartite entanglement"

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The Supplemental Material contains some additional results. We show an efficient numerical method to calculate quantities for many copies of a quantum state. We examine whether two-body interaction could be used instead of many-body interactions between the copies. We show relevant examples for the scaling of the precision with N and M. We present a simple algorithm that can be used to calculate the quantum Fisher information for the multicopy case for large systems, if the quantum Fisher information equals four times the variance. Surprisingly, there are relevant mixed states with this property. We consider error mitigation schemes based on the ideas of the paper. Finally, we present a quantum state that is maximally useful and lives in the two-copy space.

Supplement A. Efficient numerics for many copies of a quantum state

We present efficient numerical methods to obtain the quantum Fisher information for large number of copies of a bipartite state. The methods work even if a direct calculation would not be feasible.

In order to obtain the quantum Fisher information given in equation (4) for many copies, we have to compute the quantity

$$\langle k_1|\langle k_2|\langle k_3|...h_n...|l_3\rangle|l_2\rangle|l_1\rangle \tag{S1}$$

for $n = 1, 2, ..., N$ for large systems. $|l_m\rangle$ and $|k_m\rangle$ are all bipartite states, the eigenstates of the state we consider. This computation is possible if we note that if $|\Psi\rangle$ is a tensor product of two-particle states, then

$$h_n|\Psi\rangle \tag{S2}$$

is also a tensor product of two-particle states.

Let us now use the fact that the quantum Fisher information given in equation (4) can be rewritten as

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 4\langle\mathcal{H}^2\rangle - 8\sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2. \tag{S3}$$

In the following, we assume that $\lambda_k > 0$ only for $k = 1, 2, ..., r$, with r being the rank of the state ϱ.

Based on equation (S3), we can write that

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4\langle\mathcal{H}^2\rangle - 8\sum_{\vec{k},\vec{l}} \frac{\lambda_{\vec{k}} \lambda_{\vec{l}}}{\lambda_{\vec{k}} + \lambda_{\vec{l}}} |\langle k_1|\langle k_2|\langle k_3|...\mathcal{H}...|l_3\rangle|l_2\rangle|l_1\rangle|^2, \tag{S4}$$

where we use the notation

$$\lambda_{\vec{k}} = \lambda_{k_1}\lambda_{k_2}\lambda_{k_3}...\lambda_{k_M}. \tag{S5}$$

Here, states with a zero eigenvalue do not contribute. It is sufficient to look at $\vec{k}$ and $\vec{l}$ that contain only $1, 2, ..., r$. There are r^{2M} terms in the sum in equation (S4).

We can simplify further our calculations if $h_{A^{(n)}}$ are all equal to each other

$$h_{A_n^{(m)}} = h_{A_n}, \tag{S6}$$

for $n = 1, 2, ..., N$ and $m = 1, 2, ..., M$. Due to permutational invariance, only the number of different eigenvalues matter, the order does not matter. Hence, we can write that

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4\langle\mathcal{H}^2\rangle - 8\sum_{\vec{k}\in\mathcal{I}(r),\vec{l}} \mathcal{P}_{\vec{k}} \times \frac{\lambda_{\vec{k}} \lambda_{\vec{l}}}{\lambda_{\vec{k}} + \lambda_{\vec{l}}} |\langle k_1|\langle k_2|\langle k_3|...\mathcal{H}...|l_3\rangle|l_2\rangle|l_1\rangle|^2, \tag{S7}$$

where $\mathcal{I}(r)$ is the set of M-element vectors of $1, 2, ..., r$ with nondecreasing elements, the coefficients based on the number of appearance of a certain term in the sum in equation (S4) are given by the multinomial distribution

$$\mathcal{P}_{\vec{k}} = \binom{M}{m_1(\vec{k}), m_2(\vec{k}), ..., m_r(\vec{k})}. \tag{S8}$$

Here $m_l(\vec{k})$ is number of l's in $\vec{k}$.

Finally, we note that equation (S7) is a sum of positive terms that are also bounded from above. The average of positive bounded quantities can be approximated by an average obtained for a random sample.

Supplement B. Interaction between the copies via two-particle terms

Here, we consider M copies of a state of N qubits such that instead of M-particle correlations, two-particle correlations act between the copies. We show that for large number of copies, the metrological gain will be below 1 and the state is not useful. This suggests that full correlations are needed.

Let us look at the correlations

$$h_n = \sum_{m>m'} (\sigma_z)_{A_n^{(m)}} \otimes (\sigma_z)_{A_n^{(m')}} \tag{S9}$$

for $n = 1, 2, ..., N$. Then we can make a mapping for the Hamiltonian

$$\mathcal{H} \to \tilde{\mathcal{H}} = N \sum_{m>m'} \sigma_z^{(m)} \otimes \sigma_z^{(m')} \tag{S10}$$

and for the state we use the mapping given in equation (17) for $d = 2$. This way, we carry out our calculations with smaller matrices using

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = \mathcal{F}_Q[\tilde{\varrho}^{\otimes M}, \tilde{\mathcal{H}}]. \tag{S11}$$

Then, one can prove the following series of inequalities

$$\begin{aligned}\mathcal{F}_Q[\tilde{\varrho}^{\otimes M}, \tilde{\mathcal{H}}] &\leq 4N^2(\Delta M_z^2)^2_{\tilde{\varrho}^{\otimes M}} \leq 4N^2 \mathrm{Tr}(\tilde{\varrho}^{\otimes M} M_z^4) \\ &\leq 4N^2 \mathrm{Tr}(|0\rangle\langle 0|_x^{\otimes M} M_z^4) = N^2(12M^2 - 8M).\end{aligned} \tag{S12}$$

In equation (S12), the first inequality is based on that for any A the inequality $\mathcal{F}_Q[\varrho, A] \leq 4(\Delta A)^2$ holds, and that we write the sum of two-body correlations as

$$\sum_{m>m'} \sigma_z^{(m)} \otimes \sigma_z^{(m')} = (M_z^2 - M\mathbb{1})/2, \tag{S13}$$

where the z-component of the collective angular momentum is given as

$$M_z = \sum_m \sigma_z^{(m)}. \tag{S14}$$

In equation (S12), the second inequality is based on that for any A the inequality $(\Delta A)^2 \leq \langle A^2 \rangle$ holds. In equation (S12), the third inequality is based on that the product state $\tilde{\varrho}^{\otimes M}$ maximizing M_z^4 is $|0\rangle\langle 0|_x^{\otimes M}$. In equation (S12), the equality is based on simple analytical calculations.

For the best metrological performance for separable states we have

$$\mathcal{F}_Q^{(\mathrm{sep})}(\tilde{\mathcal{H}}) = N^2[\lambda_{\max}(M_z^2/2) - \lambda_{\min}(M_z^2/2)]^2. \tag{S15}$$

The largest eigenvalue is

$$\lambda_{\max}(M_z^2) = M^2. \tag{S16}$$

The smallest eigenvalue is

$$\lambda_{\min}(M_z^2) = \begin{cases} 0, & \text{if } M \text{ is even,} \\ 1, & \text{if } M \text{ is odd.} \end{cases} \tag{S17}$$

Based on these, for the metrological gain

$$g \leq \begin{cases} (48M^2 - 32M)/M^4, & \text{if } M \text{ is even,} \\ (48M^2 - 32M)/(M^2 - 1)^2, & \text{if } M \text{ is odd,} \end{cases} \tag{S18}$$

holds. The upper bound is decreasing with M rapidly. For $M \geq 7$, the metrological gain will be below 1 and the state is not useful metrologically.

Supplement C. Scaling properties

Here, we consider how the quantum Fisher information of the states appearing in Result 1 scales with the number of parties. Moreover, we also analyze quantitatively the scaling properties of the state from equation (30) in detail.

Let us consider a family of quantum states given by the same coefficients c_{kl} appearing in equation (16) for all N. For a constant M, equations (18) and (19) indicate that $\mathcal{F}_Q \propto N^2$ holds. Hence, due to the Cramér-Rao bound in equation (3), the Heisenberg scaling, $(\Delta\theta)^2 \propto 1/N^2$, can be reached. Moreover, the difference from the maximal quantum Fisher information is decreasing exponentially with M.

As a concrete example, let us consider the realistic scenario $c_{01} = \frac{1}{2}e^{-t/T}$, where T is a time constant of the decay of c_{01}. Then, from equation (21) we obtain

$$I(c_{01}, N) \approx N^2[1 - (2t/T)^{M/2}] \tag{S19}$$

for $t \ll T$, using the first two terms for the Taylor expansion of the exponential. Thus, using several copies slows down considerably the decay of the metrological abilities with t.

Now, let us consider a family of states for which the coefficients c_{kl} depend on N. In doing so, we will look at the behavior of the multicopy quantum states in the limit of large N. Let us consider the state of the type given in equation (30) for $d = 2$ and $1/N = E := 4|\sigma_0\sigma_1|^2$. Such a state has $g = 1$ metrological gain, thus it is not useful metrologically [46].

From Result 1 and equation (20), we obtain for M copies of the state

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4I_{\varrho^{\otimes M}}(\mathcal{H}) = 4N^2[1 - (1 - 1/N)^M], \tag{S20}$$

where the local Hamiltonians are

$$h_n = \sigma_z^{\otimes M}. \tag{S21}$$

For this Hamiltonian, we have $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = 4N$.

Let us consider first a constant M, not depending on N. For that case and for $N \ll M$, we obtain

$$(1 - 1/N)^M \approx 1. \tag{S22}$$

Hence, we have for the quantum Fisher information and the gain

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4N^2, \qquad g = N, \tag{S23}$$

which corresponds to the Heisenberg limit. Then, for $N \gg M$ we obtain

$$(1 - 1/N)^M \approx 1 - M/N. \tag{S24}$$

Hence, for the $N \to \infty$ limit we have for the quantum Fisher information and the gain

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4NM, \qquad g = M. \tag{S25}$$

In figure S1, it can be seen that for $N \ll M$ we reach the Heisenberg limit, while for $N \gg M$ we have at most the shot-noise scaling, *i.e.*, $(\Delta\theta)^2 \propto 1/N$.

It is instructive to look at the case that M depends on N. If we set $M = N$ then in the $N \to \infty$ limit from equation (S20) we obtain

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4N^2(1 - 1/e), \qquad g = N(1 - 1/e) \approx 0.63N, \tag{S26}$$

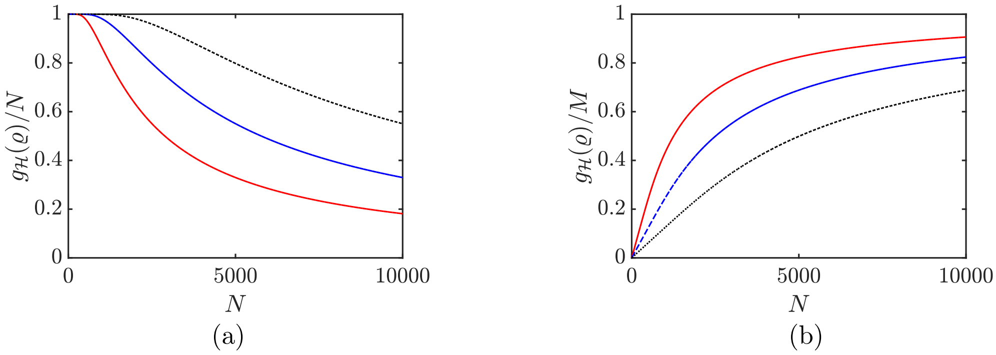

Figure S1. Dependence of the metrological gain on the particle number N for (solid) $M = 2000$, (dashed) 4000 and (dotted) 6000 copies. (a) For $N \ll M$ we have $g = N$. (b) For $N \gg M$ we have $g = M$.

where $e \approx 2.7183$ is the basis of the natural logarithm, which corresponds to the Heisenberg scaling. We used the well known relation

$$\lim_{N\to\infty} (1 - 1/N)^N = 1/e. \tag{S27}$$

If we set $N \ll M$ then in the $N \to \infty$ limit from equation (S20) we obtain

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4N^2, \qquad g = N, \tag{S28}$$

which is the Heisenberg limit. Finally, let us consider a case when the number of copies increases slower with the number of particles and set $M = \sqrt{N}$. Then, for the $N \to \infty$ limit we obtain

$$\mathcal{F}_Q[\varrho^{\otimes M}, \mathcal{H}] = 4N\sqrt{N}, \qquad g = \sqrt{N}. \tag{S29}$$

Here we used the limit

$$\lim_{N\to\infty} \frac{N^2[1 - (1 - 1/N)^{\sqrt{N}}]}{N\sqrt{N}} = 1. \tag{S30}$$

Note that the probe state we considered was weakly entangled. For states that are more entangled, we can expect that the Heisenberg scaling can be reached with $M \ll N$ copies.

Supplement D. Efficient calculations of the quantum Fisher information for large systems

Result S1. If for the eigenvectors of the quantum state

$$\langle k|\mathcal{H}|l\rangle = 0 \tag{S31}$$

holds for all k, l for which $\lambda_k > 0$ and $\lambda_l > 0$, then the quantum Fisher information can be calculated easily as

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 4(\Delta\mathcal{H})^2. \tag{S32}$$

Proof.—Evidently, we have

$$\langle \mathcal{H} \rangle = 0. \tag{S33}$$

As we have seen in equation (S3), the quantum Fisher information given in equation (4) can be rewritten as

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 4\langle \mathcal{H}^2 \rangle - 8 \sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2, \tag{S34}$$

from which the statement follows. The advantage of equation (S34) is that the summation is over cases when both λ_k and λ_l are nonzero. ■

There are bound entangled state with this property [43, 44]. This way, we can calculate the metrological performance of the state given in equation (42) for $N \geq 4$ if $\mathcal{H} = J_x$, since for that state the condition of Result S1 is fulfilled. This is true even for the multicopy case, if the local Hamiltonians are

$$h_n = \sigma_x^{\otimes M}. \tag{S35}$$

Supplement E. Error mitigation scheme

Even without additional bitflip error correcting steps, it is worth to compare our scheme to error correction in quantum metrology via a concrete example [93–95]. In the usual error correction schemes, a general pure state is stored as a fully entangled multiqubit state. Let us consider the scheme of error correction assisted metrology from [93] with a bitflip code such that a logical qubit corresponds to three physical qubits. Then we have the state

$$\frac{1}{\sqrt{2}}(|000\ 000\ 000\rangle + |111\ 111\ 111\rangle) \tag{S36}$$

and the Hamiltonian

$$H = \sigma_z^{(1)}\sigma_z^{(2)}\sigma_z^{(3)} + \sigma_z^{(4)}\sigma_z^{(5)}\sigma_z^{(6)} + \sigma_z^{(7)}\sigma_z^{(8)}\sigma_z^{(9)}.$$

For this case we also need error syndrome measurements to detect the error. Using the phase-flip code, we can suppress phase errors similarly, only a basis transformation is needed.

In contrast, in the case of multicopy metrology, we have M copies of the state that are not entangled to each other initially and will become slightly entangled during the metrology. Let us consider the $M = 3$ case with $h_n = \sigma_z^{\otimes 3}$. Then, the state of the system for $N = 3$ is

$$|\mathrm{GHZ}\rangle^{\otimes 3} = [\tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle)]^{\otimes 3}, \tag{S37}$$

and the Hamiltonian is

$$\mathcal{H} = \sigma_z^{(1)}\sigma_z^{(4)}\sigma_z^{(7)} + \sigma_z^{(2)}\sigma_z^{(5)}\sigma_z^{(8)} + \sigma_z^{(3)}\sigma_z^{(6)}\sigma_z^{(9)}.$$

Our approach improves the metrological performance without syndrome measurements. Moreover, it can suppress the effects of phase errors. In particular, the following result holds for $N \geq 2$ and $M = 3$.

Result S2. Let ϱ be a mixture of the following states that have phase error on at most one copy

$$\begin{aligned} |\Psi_0\rangle \quad &= |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \\ |\Psi_{1,\phi_1}\rangle &= |\mathrm{GHZ}_{\phi_1}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \\ |\Psi_{2,\phi_2}\rangle &= |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_2}\rangle \otimes |\mathrm{GHZ}\rangle, \\ |\Psi_{3,\phi_3}\rangle &= |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_3}\rangle, \end{aligned} \tag{S38}$$

where

$$|\mathrm{GHZ}_\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + e^{-i\phi}|1\rangle^{\otimes N}),$$

and ϕ_1, ϕ_2 and ϕ_3 are arbitrary phase factors. We still obtain the maximal

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 4N^2,$$

where $\mathcal{H}$ is given by equation (2), such that $h_n = \sigma_z^{\otimes 3}$. For $N = 3$, the optimal operator to be measured is

$$\sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes (\sigma_y)^{\otimes 6} + (\sigma_y)^{\otimes 6} \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} + (\sigma_y)^{\otimes 3} \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1} \otimes (\sigma_y)^{\otimes 3} + \sigma_y^{\otimes 9}. \tag{S39}$$

Proof. Let us write ϱ as

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|, \tag{S40}$$

where $|\Psi_k\rangle$'s are chosen from the set in equation (S38). Then, for any k, l we have

$$\langle\Psi_k|\mathcal{H}|\Psi_l\rangle = 0. \tag{S41}$$

This can be seen as follows. Let us consider the tensor product form of $|\Psi_k\rangle$ from equation (S38) as

$$|\Psi_k\rangle = \otimes_{m=1}^{3} |\Psi_{k,m}\rangle, \tag{S42}$$

where $|\Psi_{k,m}\rangle$'s are single copy states. We have

$$\langle h_n\rangle = \prod_{m=1}^{M} \langle\Psi_{k,m}|\sigma_z|\Psi_{l,m}\rangle. \tag{S43}$$

There is at most one erroneous copy in $|\Psi_k\rangle$ so for any k, l there is always an m such that $|\Psi_{k,m}\rangle = |\Psi_{l,m}\rangle = |\mathrm{GHZ}\rangle$. With this m, the relation $\langle\Psi_{k,m}|\sigma_z|\Psi_{l,m}\rangle = 0$ holds. Hence,

$$\langle\Psi_k|\mathcal{H}|\Psi_l\rangle = \left\langle \Psi_k \middle| \sum_n h_n \middle| \Psi_l \right\rangle = 0. \tag{S44}$$

Let us now consider the eigendecomposition of ϱ

$$\varrho = \sum_k \lambda_k |k\rangle\langle k|. \tag{S45}$$

Then, any eigenvector for which $\lambda_k > 0$ holds can be obtained as

$$|k\rangle = \sum_l c_{kl} |\Psi_l\rangle, \tag{S46}$$

where c_{kl} are some constants. Hence, for the eigenvectors in the range of ϱ we have

$$\langle k|\mathcal{H}|l\rangle = 0. \tag{S47}$$

Hence, it follows that [43, 44]

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 4(\Delta\mathcal{H})^2_\varrho = 4\langle\mathcal{H}^2\rangle_\varrho. \tag{S48}$$

Note that an optimization aiming at maximizing the quantum Fisher information within some set of quantum states often results in states satisfying equation (S48) [43, 44]. Based on equation (S48), due to the convexity of the quantum Fisher information and the concavity of the variance follows that for any $|\Psi_k\rangle$ having the properties mentioned above

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = \sum_k p_k \mathcal{F}_Q[\Psi_k, \mathcal{H}] \tag{S49}$$

holds. If $\mathcal{F}_Q[\Psi_k, \mathcal{H}]$ are maximal, so is $\mathcal{F}_Q[\varrho, \mathcal{H}]$. Straightforward algebra shows that the operator given in equation (S39) provides a precision that saturates the Cramér-Rao bound. ■

Analogous arguments show that for any odd $M > 3$ we can allow at most $(M-1)/2$ copies with phase errors and the quantum Fisher information will still be maximal. This is true even if we consider a mixture of the above states.

Note that a minimal experimental test is also possible with only two copies. In this case, we can consider a state that has no error and states that have an error on the first copy.

Note also that the application of the Hamiltonians $h_n = \sigma_z^{\otimes M}$ can be replaced by applying phase gates before and after the dynamics and using the Hamiltonian $h_n = \sigma_z \otimes \mathbb{1}^{\otimes(M-1)}$ [93]. In this case, we can assume that the phase noise acts on the qubit that is coupled by σ_z.

Supplement F. State living in the two-copy space

In this section, we present a quantum state living in the two-copy space of a bipartite system. We denote the two copies AB and $A'B'$.

Let us consider the quantum state

$$\varrho_p = p|\Psi^+\rangle\langle\Psi^+|^{\otimes 2} + (1-p)|\Phi^+\rangle\langle\Phi^+|^{\otimes 2}, \tag{S50}$$

where $|\Phi^+\rangle$ is defined as in equation (48), and

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \tag{S51}$$

For state ϱ_p and the Hamiltonian

$$H = \sigma_z^A \sigma_z^{A'} + \sigma_z^B \sigma_z^{B'} \tag{S52}$$

for any p the metrological gain is obtained as

$$g = 2. \tag{S53}$$

The state ϱ_p given in equation (S50) is equivalent to the state that is the mixture of the maximally entangled state

$$\frac{1}{2}\sum_{n=0}^{3}|n\rangle_{AA'}|n\rangle_{BB'}, \tag{S54}$$

and state obtained from it after a basis transformation

$$\frac{1}{2}\sum_{n=0}^{3}|n\rangle_{AA'}|3-n\rangle_{BB'}. \tag{S55}$$

It is instructive to compare the state given in equation (S50) to

$$\varrho_p = p|\Psi^+\rangle\langle\Psi^+| + (1-p)|\Phi^+\rangle\langle\Phi^+|, \tag{S56}$$

which is nonentangled and metrologically not useful for $p = 1/2$.