Macroscopic singlet states for gradient magnetometry (Supplementary Material)

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In the supplement, we present additional calculations for obtaining the accuracy of the gradient measurement.

I. ADDITIONAL CALCULATIONS FOR OBTAINING $\langle J_x^4\rangle_{\rm s}(\Theta)$ FROM OBSERVATION 3

The general expression of $\langle J_x^4\rangle_{\rm s}(\Theta)$ for N particles being at the positions $z_k^{\rm c}$, collected in a vector $\vec{z}_N^{\,\rm c}$, has been obtained in Eq. (59). Here and in the following we will label with $\vec{z}_N^{\,\rm c}$ a vector with N elements $z_1^{\rm c},...,z_N^{\rm c}$. In order to being able to compute this for large N, we need to simplify

$$I_4 := I_{\rm cccc} + I_{\rm ssss} + 2I_{\rm ccss}, \tag{S1}$$

where

$$\begin{aligned} I_{\rm cccc} &:= \sum_{\neq(k,l,m,n)} c_k c_l c_m c_n, \\ I_{\rm ssss} &:= \sum_{\neq(k,l,m,n)} s_k s_l s_m s_n, \\ I_{\rm ccss} &:= \sum_{\neq(k,l,m,n)} c_k c_l s_m s_n, \end{aligned} \tag{S2}$$

where $c_k = \cos(\frac{z_k^{\rm c}}{L}\Theta)$ and $s_k = \sin(\frac{z_k^{\rm c}}{L}\Theta)$. We will show two ways of doing this. Firstly, as stated in Eq. (60), one can rewrite it as

$$\begin{aligned} I_4 \;=\; & X_{1,0}^4 + X_{0,1}^4 + 2X_{1,0}^2X_{0,1}^2 \\ & -(6X_{2,0}X_{1,0}^2 + 6X_{0,2}X_{0,1}^2 + 2X_{2,0}X_{0,1}^2 + 2X_{0,2}X_{1,0}^2 + 8X_{1,1}X_{1,0}X_{0,1}) \\ & +8X_{3,0}X_{1,0} + 3X_{2,0}^2 + 8X_{0,3}X_{0,1} + 3X_{0,2}^2 + 8X_{2,1}X_{0,1} + 2X_{2,0}X_{0,2} + 8X_{1,2}X_{1,0} + 4X_{1,1}^2 \\ & -(6X_{4,0} + 6X_{0,4} + 12X_{2,2}), \end{aligned} \tag{S3}$$

where

$$X_{m,n} := \sum_{k=1}^{N} c_k^m s_k^n. \tag{S4}$$

The proof is presented in Section I A below. Secondly, as stated in Eq. (62), one may also rewrite it more compactly as

$$I_4 = N\left\{2(N-3) - 4N(N-2)\left|\hat{f}_1(\alpha)\right|^2 + N^3\left|\hat{f}_1(\alpha)\right|^4 + N\left|\hat{f}_1(2\alpha)\right|^2 - 2N^2{\rm Re}\big[\hat{f}_1^2(\alpha)\hat{f}_1(2\alpha)^*\big]\right\}, \tag{S5}$$

where

$$\hat{f}_1(\alpha) = \frac{1}{N}\sum_k e^{i\alpha z_k^{\rm c}} \quad \text{and} \quad \alpha = \frac{\Theta}{L}. \tag{S6}$$

The proof is presented in Section I B below.

A. Proof of Eq. (S3)

Let us concentrate on the last term in Eq. (S1). We can write using a shorthand notation

$$
\begin{aligned}
I_{\rm ccss} = \Bigg(& \sum_{k,l,m,n} \\
& - \sum_{\neq(k=l,m,n)} - \sum_{\neq(k,l,m=n)} - \sum_{\neq(k=m,l,n)} - \sum_{\neq(k=n,m,n)} - \sum_{\neq(k,l=m,n)} - \sum_{\neq(k,l=n,m)} \\
& - \sum_{\neq(k=l,m=n)} - \sum_{\neq(k=m,l=n)} - \sum_{\neq(k=n,m=n)} \\
& - \sum_{\neq(k=l=m,n)} - \sum_{\neq(k=l=n,m)} - \sum_{\neq(k=m=n,l)} - \sum_{\neq(l=m=n,k)} \\
& - \sum_{k=l=m=n} \Bigg) c_k c_l s_m s_n. \qquad \text{(S7)}
\end{aligned}
$$

Here $\neq (k=m, l=n)$ means that the summation is such that $k=m, l=n$ and $k \neq l$. Eq. (S7) can be rewritten after simple considerations as

$$
\begin{aligned}
I_{\rm ccss} &= \sum_{k,l,m,n} c_k c_l s_m s_n - \sum_{\neq(k,l,m)} (c_k^2 s_l s_m + c_k c_l s_m^2 + 4 c_k s_k c_l s_m) \\
&\quad - \sum_{\neq(k,l)} (c_k^2 s_l^2 + 2 c_k s_k c_l s_l + 2 c_k^2 s_k c_l + 2 c_k c_l s_l^2) - \sum_k c_k^2 s_k^2. \qquad \text{(S8)}
\end{aligned}
$$

Each term in Eq. (S8) corresponds to a line in Eq. (S7). Then, we can rewrite the terms in Eq. (S8) still containing the conditions "not equal" with terms without such conditions as follows

$$
\begin{aligned}
\sum_{\neq(k,l,m)} c_k^2 s_l s_m &= \Bigg(\sum_{k,l,m} - \sum_{\neq(k=l,m)} - \sum_{\neq(k,l=m)} - \sum_{\neq(k=m,l)} - \sum_{k=m=n} \Bigg) c_k^2 s_l s_m \\
&= \sum_{k,l,m} c_k^2 s_l s_m \\
&\quad - 2 (\sum_{k,l} c_k^2 s_k s_l - \sum_k c_k^2 s_k^2) - (\sum_{k,l} c_k^2 s_l^2 - \sum_k c_k^2 s_k^2) \\
&\quad - \sum_k c_k^2 s_k^2, \\
&= \sum_{k,l,m} c_k^2 s_l s_m - \sum_{k,l} (2 c_k^2 s_k s_l + c_k^2 s_l^2) + 2 \sum_k c_k^2 s_k^2, \qquad \text{(S9)}
\end{aligned}
$$

where we used that we have

$$
\sum_{\neq(k,l)} a_k b_l = \sum_{k,l} a_k b_l - \sum_k a_k b_k \qquad \text{(S10)}
$$

for any real numbers a_k and b_k. Analogously, one finds that

$$
\sum_{\neq(k,l,m)} c_k c_l s_m^2 = \sum_{k,l,m} c_k c_l s_m^2 - \sum_{k,l} (2 c_k c_l s_l^2 + c_k^2 s_l^2) + 2 \sum_k c_k^2 s_k^2, \qquad \text{(S11)}
$$

and

$$
\sum_{\neq(k,l,m)} c_k s_k c_l s_m = \sum_{k,l,m} c_k s_k c_l s_m - \sum_{k,l} (c_k^2 s_k s_l + c_k s_k c_l s_l + c_k c_l s_l^2) + 2 \sum_k c_k^2 s_k^2. \qquad \text{(S12)}
$$

Substituting Eqs. (S9), (S11), and (S12) into Eq. (S8), and using again Eq. (S10) for the remaining terms of two non-equal indices, we arrive at

$$\begin{aligned} I_{\rm ccss} = & \sum_{k,l,m,n} c_k c_l s_m s_n \\ & - \sum_{k,l,m} c_k^2 s_l s_m + \sum_{k,l} (2c_k^2 s_k s_l + c_k^2 s_l^2) - 2\sum_k c_k^2 s_k^2 \\ & - \sum_{k,l,m} c_k c_l s_m^2 + \sum_{k,l} (2c_k c_l s_l^2 + c_k^2 s_l^2) - 2\sum_k c_k^2 s_k^2 \\ & - 4\sum_{k,l,m} c_k s_k c_l s_m + 4\sum_{k,l} (c_k^2 s_k s_l + c_k s_k c_l s_l + c_k c_l s_l^2) - 8\sum_k c_k^2 s_k^2 \\ & - \sum_{k,l} (c_k^2 s_l^2 + 2c_k s_k c_l s_l + 2c_k^2 s_k s_l + 2c_k c_l s_l^2) + 7\sum_k c_k^2 s_k^2 \\ & - \sum c_k^2 s_k^2. \end{aligned} \tag{S13}$$

In Eq. (S13), the first four lines correspond to the first line in Eq. (S8), and the remaining two lines to the second line. This can be simplified by combining terms that appear more than once as

$$\begin{aligned} I_{\rm ccss} = & \sum_{k,l,m,n} c_k c_l s_m s_n \\ & - \sum_{k,l,m} \left(c_k^2 s_l s_m + c_k c_l s_m^2 + 4c_k s_k c_l s_m \right) \\ & + \sum_{k,l} \left(c_k^2 s_k s_l (2+4-2) + c_k^2 s_l^2 (1+1-1) + c_k c_l s_l^2 (2+4-2) + c_k s_k c_l s_l (4-2) \right) \\ & + \sum_k c_k^2 s_k^2 (-2-2-8+7-1), \end{aligned} \tag{S14}$$

which finally yields

$$\begin{aligned} I_{\rm ccss} \ = \ & \sum_{k,l,m,n} c_k c_l s_m s_n - \sum_{k,l,m} \left(c_k^2 s_l s_m + c_k c_l s_m^2 + 4c_k s_k c_l s_m \right) \\ & + \sum_{k,l} \left(4c_k^2 s_k s_l + c_k^2 s_l^2 + 4c_k c_l s_l^2 + 2c_k s_k c_l s_l \right) - 6\sum_k c_k^2 s_k^2. \end{aligned} \tag{S15}$$

The formula for $I_{\rm cccc}$ can be obtained from the formula for $I_{\rm ccss}$ [Eq. (S15)], by replacing s_k by c_k and combining terms that appear more than once as

$$\begin{aligned} I_{\rm cccc} &= \sum_{k,l,m,n} c_k c_l c_m c_n - \sum_{k,l,m} c_k^2 c_l c_m (1+1+4) + \sum_{k,l} c_k^3 c_l (4+4) + \sum_{k,l} c_k^2 c_l^2 (1+2) - 6\sum_k c_k^4 \\ &= \sum_{k,l,m,n} c_k c_l c_m c_n - 6\sum c_k^2 c_l c_m + \sum_{k,l} (8c_k^3 c_l + 3c_k^2 c_l^2) - 6\sum_k c_k^4. \end{aligned} \tag{S16}$$

Similarly, the formula for $I_{\rm ssss}$ is obtained as

$$I_{\rm ssss} = \sum_{k,l,m,n} s_k s_l s_m s_n - 6\sum_{k,l,m} s_k^2 s_l s_m + \sum_{k,l} (8s_k^3 s_l + 3s_k^2 s_m^2) - 6\sum_k s_k^4. \tag{S17}$$

Combining the results of the Eqs. (S15), (S16), and (S17), we obtain

$$
\begin{aligned}
I_4 = \; & I_{\rm cccc} + I_{\rm ssss} + 2I_{\rm ccss} \\
= \; & \sum_{k,l,m,n} c_k c_l c_m c_n + s_k s_l s_m s_n + 2 c_k c_l s_m s_n \\
& - \sum_{k,l,m} \left(6c_k^2 c_l c_m + 6s_k^2 s_l s_m + 2c_k^2 s_l s_m + 2c_k c_l s_m^2 + 8c_k s_k c_l s_m\right) \\
& + \sum_{k,l} \left(8c_k^3 c_l + 3c_k^2 c_l^2 + 8s_k^3 s_l + 3s_k^2 s_m^2 + 8c_k^2 s_k s_l + 2c_k^2 s_l^2 + 8c_k c_l s_l^2 + 4c_k s_k c_l s_l\right) \\
& -6 \sum_k (c_k^4 + s_k^4 + 2c_k^2 s_k^2). \qquad \text{(S18)}
\end{aligned}
$$

This is equivalent to Eq. (S3). □

B. Proof of Eq. (S5)

Using the continuous distribution formalism one can write I_4 from Eq. (S1) as

$$
I_4 = \frac{N!}{(N-4)!} \int {\rm d}\vec{z}_4 f_4^{\vec{z}_N^{\rm c}}(\vec{z}_4) \Big(c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4 + 2 c_1 c_2 s_3 s_4 \Big), \qquad \text{(S19)}
$$

where $f_4^{\vec{z}_N^{\rm c}}(\vec{z}_4)$ is the reduced 4-body correlation function for the chain, cf. Eq. (72) of the main text. It is computed as $f_4^{\vec{z}_N^{\rm c}}(\vec{z}_4) = \int {\rm d}z_5 \cdots {\rm d}z_N f_N^{\vec{z}_N^{\rm c}}(\vec{z}_N)$ from the permutationally invariant N-variate probability density of N particles with the z-coordinates $\vec{z}_N^{\,\rm c}$, which is given by

$$
f_N^{\vec{z}_N^{\rm c}}(\vec{z}_N) = \frac{1}{N!} \sum_{\pi \in S_N} \prod_{k=1}^{N} \delta\left(z_k - z_{\pi(k)}^{\rm c}\right) = \frac{1}{N!} \sum_{\neq (k_1, k_2, \dots, k_N)} \prod_{j=1}^{N} \delta\left(z_j - z_{k_j}^{\rm c}\right). \qquad \text{(S20)}
$$

Here, S_N is the permutation group of N particles and the first sum runs over all permutations π from that group. In the second sum the k_j indices ranges from 1 to N with the restriction that the indices be different, and $z_k^{\rm c}$ are the locations of the particles on the chain.

For *any* permutationally invariant probability density f_N one can show that

$$
\int {\rm d}\vec{z}_4 f_4 \Big(c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4 + 2 c_1 c_2 s_3 s_4 \Big) = \int {\rm d}\vec{z}_4 f_4 \cos\left[\frac{z_1 - z_2 + z_3 - z_4}{L}\Theta\right] = \int {\rm d}\vec{z}_4 f_4 e^{i \frac{z_1 - z_2 + z_3 - z_4}{L}\Theta} \qquad \text{(S21)}
$$

holds. The second equality holds because the sine expression occuring in the exponent is an odd function under the exchange of $z_1 + z_3$ and $z_2 + z_4$. In this way, we can express I_4 with the help of characteristic functions. In particular, the multivariate characteristic of the multivariate probability density $f_N(\vec{z}_N)$ is

$$
\hat{f}_N(\vec{\alpha}_N) = \left\langle e^{i \sum_{k=1}^N \alpha_k z_k} \right\rangle = \int {\rm d}\vec{z}_N f_N(\vec{z}_N) e^{i \sum_{k=1}^N \alpha_k z_k}. \qquad \text{(S22)}
$$

As can be easily checked, $\hat{f}_N(\vec{\alpha}_N)$ has the following properties: (i) $\hat{f}_N$ is permutationally invariant if f_N is, (ii) $\hat{f}_N\Big([\alpha_1, \alpha_2, ..., \alpha_{N-1}, 0]\Big) = \hat{f}_{N-1}(\vec{\alpha}_{N-1})$, (iii) $\hat{f}_N(\vec{0}_N) = 1$, where $\vec{0}_N$ is a vector where all entries are equal to 0, and (iv) $f_N(-\vec{\alpha}_N) = f_N(\vec{\alpha}_N)^*$. Property (ii) follows from the fact that $\int {\rm d}z_N f_N(\vec{z}_N) = f_{N-1}(\vec{z}_{N-1})$ and property (iii) follows from the normalization of $f_N(\vec{z}_N)$.

Comparing the Equations (56), (S21), and (S22), we observe that

$$
I_4 = \frac{N!}{(N-4)!} \hat{f}_4^{\vec{z}_N^{\rm c}}(\alpha, -\alpha, \alpha, -\alpha), \quad \text{where} \quad \alpha = \frac{\Theta}{L}, \qquad \text{(S23)}
$$

and where $\hat{f}_4^{\vec{z}_N^{\rm c}}$ has to be computed from the probability density of Eq. (S20). In general, the lower elements $f_M^{\vec{z}_N^{\rm c}}(\vec{z}_M)$ ($M \leq N$) are given by

$$
f_M^{\vec{z}_N^{\rm c}}(\vec{z}_M) = \frac{(N-M)!}{N!} \sum_{\neq (k_1, k_2, \dots, k_M)} \prod_{j=1}^{M} \delta\left(z_j - z_{k_j}^{\rm c}\right). \qquad \text{(S24)}
$$

Let us compute the characteristic function of $f_M^{\vec{z}_N^{\rm c}}$, dropping from now on the upper index $\vec{z}_N^{\rm c}$ in order to simplify the notation. We obtain the following recurrence relation

$$\begin{aligned}
\hat{f}_M(\vec{\alpha}_M) &= \frac{(N-M)!}{N!} \sum_{\neq(k_1,k_2,\ldots,k_M)} e^{i\sum_{j=1}^{M}\alpha_j z^0_{k_j}} \\
&= \frac{(N-M)!}{N!} \left\{ \sum_{\neq(k_1,k_2,\ldots,k_{M-1}),k_M} e^{i\sum_{j=1}^{M}\alpha_j z^0_{k_j}} - \sum_{\neq(k_1,k_2,\ldots,k_{M-1})} \sum_{l=1}^{M-1} e^{i\sum_{j=1}^{M-1}\alpha_j z^0_{k_j}} e^{i\alpha_M z^0_{k_l}} \right\} \\
&= \frac{(N-M)!}{N!} \left\{ N\left\langle e^{i\alpha_M z_1}\right\rangle \frac{N!}{(N-M+1)!} \left\langle e^{i\sum_{j=1}^{M-1}\alpha_j z^0_j}\right\rangle - \frac{N!}{(N-M+1)!} \sum_{l=1}^{M-1} \left\langle e^{i\sum_{j=1}^{M-1}\alpha_j z^0_j} e^{i\alpha_M z^0_l}\right\rangle \right\} \\
&= \frac{1}{N-M+1} \left\{ N\hat{f}_1\left(\alpha_M\right)\hat{f}_{M-1}(\vec{\alpha}_{M-1}) - \sum_{l=1}^{M-1} \hat{f}_{M-1}(\vec{\alpha}_{M-1}+\alpha_M\hat{e}_l) \right\}, \qquad \text{(S25)}
\end{aligned}$$

where $\hat{e}_l$ is a vector of length $M-1$ that has only one nonvanishing element (that is equal to 1) at the position l. It can be used to compute I_4 *via* Eq. (S23), leading to

$$\hat{f}_4(\alpha,-\alpha,\alpha,-\alpha) = \frac{1}{N-3}\left\{N\hat{f}_1^*(\alpha)\hat{f}_3(\alpha,-\alpha,\alpha) - 2\hat{f}_2(\alpha,-\alpha) + \hat{f}_3(\alpha,-2\alpha,\alpha)\right\} \qquad \text{(S26)}$$

We can apply the recurrence relation again for $M=3$ in order to reduce the complexity of this expression. We obtain

$$\begin{aligned}
\hat{f}_3(\vec{\alpha}_3) &= \frac{1}{(N-1)(N-2)}\Big\{N^2\hat{f}_1(\alpha_1)\hat{f}_1(\alpha_2)\hat{f}_1(\alpha_3) + 2\hat{f}_1(\alpha_1+\alpha_2+\alpha_3) \\
&\quad N\big[\hat{f}_1(\alpha_1)\hat{f}_1(\alpha_2+\alpha_3) + \hat{f}_1(\alpha_2)\hat{f}_1(\alpha_1+\alpha_3) + \hat{f}_1(\alpha_3)\hat{f}_1(\alpha_1+\alpha_2)\big]\Big\}\,,
\end{aligned}$$

which for the two cases of interest in Eq. (S26) yields

$$\begin{aligned}
\hat{f}_3(\alpha,-\alpha,\alpha) &= \frac{1}{(N-1)(N-2)}\left\{N^2\left|\hat{f}_1(\alpha)\right|^2\hat{f}_1(\alpha) - N\hat{f}_1(2\alpha)\hat{f}_1^*(\alpha) - 2(N-1)\hat{f}_1(\alpha)\right\}\,, \\
\hat{f}_3(\alpha,-2\alpha,\alpha) &= \frac{1}{(N-1)(N-2)}\left\{N^2\hat{f}_1^2(\alpha)\hat{f}_1^*(2\alpha) - 2N\left|\hat{f}_1(\alpha)\right|^2 - N\left|\hat{f}_1(2\alpha)\right|^2 + 2\right\}\,.
\end{aligned}$$

Similarly, we obtain for $M=2$ that

$$\hat{f}_2\left(\alpha_1,\alpha_2\right) = \frac{1}{N-1}\left\{N\hat{f}_1\left(\alpha_1\right)\hat{f}_1\left(\alpha_2\right) - \hat{f}_1\left(\alpha_1+\alpha_2\right)\right\} \qquad \text{(S27)}$$

For the special case of interest $\alpha_1 = -\alpha_2$ occuring in Eq. (S26) this reduces to

$$\hat{f}_2\left(\alpha,-\alpha\right) = \frac{1}{N-1}\left\{N\left|\hat{f}_1\left(\alpha\right)\right|^2 - 1\right\}\,. \qquad \text{(S28)}$$

Finally

$$\begin{aligned}
\hat{f}_4(\alpha,-\alpha,\alpha,-\alpha) &= \frac{1}{(N-1)(N-2)(N-3)}\Big\{2(N-3) - 4N(N-2)\left|\hat{f}_1(\alpha)\right|^2 + N^3\left|\hat{f}_1(\alpha)\right|^4 + N\left|\hat{f}_1(2\alpha)\right|^2 \\
&\qquad -2N^2\mathrm{Re}\big[\hat{f}_1^2(\alpha)\hat{f}_1(2\alpha)^*\big]\Big\}\,,
\end{aligned}$$

which due to the identity (S23) is equivalent to Eq. (S5) for $\hat{f}_1(\alpha) = \frac{1}{N}\sum_k e^{i\alpha z_k^{\rm c}}$ [computed with the Eqs. (S22) and (S24)] with $\alpha = \frac{\Theta}{L}$. □

II. ADDITIONAL CALCULATIONS FOR OBTAINING $(\Delta\Theta)_{\rm s}^{-2}|_{\Theta=0}$ FOR OBSERVATION 7

We will show that for $\Theta \to 0$, the inverse variance of the estimation of Θ is given by

$$(\Delta\Theta)_{\rm s}^{-2}|_{\Theta=0} = \frac{N}{L^2}\left[\sigma^2 - \mathrm{cov}(z_1,z_2)\right], \tag{S29}$$

where

$$\begin{aligned}
\sigma^2 &= \int \mathrm{d}z_1 f_1(z_1)(z_1 - \langle z_1\rangle)^2,\\
\langle z_1\rangle &= \int \mathrm{d}z_1 f_1(z_1) z_1,\\
\mathrm{cov}(z_1,z_2) &= \int \mathrm{d}z_1 \mathrm{d}z_2 f_2(z_1,z_2)(z_1-\langle z_1\rangle)(z_2-\langle z_2\rangle) = \langle z_1 z_2\rangle - \langle z_1\rangle\langle z_2\rangle.
\end{aligned} \tag{S30}$$

Proof. We estimate the uncertainty from the error propagation formula

$$(\Delta\Theta)_{\rm s}^2 = \frac{(\Delta J_x^2)_{\rm s}}{|\partial_\Theta \langle J_x^2\rangle_{\rm s}|^2}, \tag{S31}$$

cf. Eq. (48) of the main text, for general continuous density profiles. The quantities which occur are $\langle J_x^2\rangle_{\rm s}(\Theta)$, $\partial_\Theta\langle J_x^2\rangle_{\rm s}(\Theta)$ and $\langle J_x^4\rangle_{\rm s}(\Theta)$. In order to get the desired limit, we need to expand them around $\Theta = 0$. Let us start with $\langle J_x^2\rangle_{\rm s}(\Theta)$. For fixed particle positions $\vec{z}_N$, we obtain

$$\langle J_x^2\rangle_{\rm s}^{\vec{z}_N}(\Theta) = \frac{N\hbar^2}{4}\Big[1 - \frac{1}{N(N-1)}\sum_{n\neq m}\cos\Big(\frac{z_n - z_m}{L}\Theta\Big)\Big],$$

from the Eqs. (37,38) and (43). Averaging this over a general permutationally independent density profile $f_N(\vec{z}_N)$, we obtain

$$\langle J_x^2\rangle_{\rm s}^{f_N}(\Theta) = \frac{N\hbar^2}{4}\left[1 - \int \mathrm{d}z_1\mathrm{d}z_2 f_2(z_1,z_2)\cos\Big(\frac{z_1-z_2}{L}\Theta\Big)\right] \equiv \frac{N\hbar^2}{4}(1-\tilde{I}_2).$$

Expanding the cosine in the integral we arrive at

$$\begin{aligned}
\tilde{I}_2 &\approx 1 - \frac{1}{2L^2}\int \mathrm{d}z_1\mathrm{d}z_2 f_2(z_1,z_2)(z_1-z_2)^2\ \Theta^2 + O(\Theta^4)\\
&= 1 - \frac{1}{L^2}\Big(\int \mathrm{d}z_1 f_1(z_1) z_1^2 - \int \mathrm{d}z_1\mathrm{d}z_2 f_2(z_1,z_2) z_1 z_2\Big)\Theta^2 + O(\Theta^4)\\
&= 1 - \frac{1}{L^2}\Big(\sigma^2 - \mathrm{cov}(z_1,z_2)\Big)\Theta^2 + O(\Theta^4),
\end{aligned} \tag{S32}$$

where the last line is obtained by adding and subtracting a term $\langle z_1\rangle^2$. We also used that due to the permutational invariance $\langle z_1^2\rangle = \langle z_2^2\rangle$ and $\langle z_1\rangle = \langle z_2\rangle$ hold. This leads to

$$\langle J_x^2\rangle_{\rm s}^{f_N}(\Theta) \approx \frac{N\hbar^2}{4L^2}\Big(\sigma^2 - \mathrm{cov}(z_1,z_2)\Big)\Theta^2 + O(\Theta^4) \tag{S33}$$

and

$$\partial_\Theta\langle J_x^2\rangle_{\rm s}^{f_N}(\Theta) \approx \frac{N\hbar^2}{2L^2}\Big(\sigma^2 - \mathrm{cov}(z_1,z_2)\Big)\Theta + O(\Theta^3), \tag{S34}$$

Let us now consider the expansion of the term $\langle J_x^4\rangle_{\rm s}(\Theta)$. Again for fixed positions $\vec{z}_N$, we have [cf. Eq. (67)]

$$\frac{\langle J_x^4\rangle_{\rm s}^{\vec{z}_N}(\Theta)}{\hbar^4} = \frac{3N^2-2N}{16} - \frac{3N-4}{8(N-1)}\sum_{k\neq l}\cos\Big(\frac{z_n-z_m}{L}\Theta\Big) + \frac{3}{16}\frac{1}{(N-1)(N-3)}I_4,$$

with I_4 from Eq. (S1) above. Note that in contrast to this equation, the particle positions are labelled by $\vec{z}_N$ instead of $\vec{z}_N^{\,\rm c}$ because we have to average the expression over f_N. This leads to

$$\begin{aligned}
\frac{\langle J_x^4\rangle_{\rm s}^{f_N}(\Theta)}{\hbar^4} &= \frac{3N^2-2N}{16} - \frac{N(3N-4)}{8}\int \mathrm{d}z_1\mathrm{d}z_2\ f_2(z_1,z_2)\ \cos\Big(\frac{z_1-z_2}{L}\Theta\Big)\\
&+ \frac{3N(N-2)}{16}\int \mathrm{d}^4z f_4(\vec{z}_4)\Big(c_1c_2c_3c_4 + s_1s_2s_3s_4 + 2c_1c_2s_3s_4\Big)\\
&\equiv \frac{3N^2-2N}{16} - \frac{N(3N-4)}{8}\tilde{I}_2 + \frac{3N(N-2)}{16}\tilde{I}_4,
\end{aligned} \tag{S35}$$

where we used again the permutational invariance of f_N. We need to expand the expression $\tilde{I}_4$. Using the first equality from Eq. (S21) and expanding the occurring cosine as before one obtains that

$$\begin{aligned}\tilde{I}_4 &\approx 1 - \frac{1}{2L^2}\int \mathrm{d}^4 z f_4(z_1,z_2,z_3,z_4)(z_1+z_2-z_3-z_4)^2\Theta^2 + O(\Theta^4)\\ &= 1 - \frac{2}{L^2}[\sigma^2 - \mathrm{cov}(z_1,z_2)]\Theta^2 + O(\Theta^4).\end{aligned} \tag{S36}$$

Inserting the expansions of $\tilde{I}_2$ from Eq. (S32) and of $\tilde{I}_4$ from Eq. (S36) into Eq. (S35) leads to

$$\begin{aligned}\frac{\langle J_x^4\rangle_{\mathrm{s}}^{f_N}(\Theta)}{\hbar^4} &\approx \frac{3N^2-2N}{16} - \frac{N(3N-4)}{8}\Big(1 - \frac{1}{L^2}[\sigma^2 - \mathrm{cov}(z_1,z_2)]\Theta^2\Big)\\ &+ \frac{3N(N-2)}{16}\Big(1 - \frac{2}{L^2}[\sigma^2 - \mathrm{cov}(z_1,z_2)]\Theta^2\Big) + O(\Theta^4)\\ &= \frac{N}{4L^2}[\sigma^2 - \mathrm{cov}(z_1,z_2)]\Theta^2 + O(\Theta^4).\end{aligned} \tag{S37}$$

Now we have all the necessary ingredients to prove the claim. Indeed, inserting the Eqs. (S33,S34,S37) into Eq. (S31) we obtain

$$(\Delta\Theta)_{\mathrm{s}}^{-2} \approx \frac{\frac{N^2\hbar^4}{4L^4}[\sigma^2 - \mathrm{cov}(z_1,z_2)]^2\Theta^2 + O(\Theta^4)}{\frac{N\hbar^4}{4L^2}[\sigma^2 - \mathrm{cov}(z_1,z_2)]\Theta^2 + O(\Theta^4)} = \frac{N}{L^2}[\sigma^2 - \mathrm{cov}(z_1,z_2)] + O(\Theta^2),$$

which proves Eq. (S29). □