Spin squeezing inequalities for arbitrary spin - Supplementary material

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The supplement contains some derivations to help to understand the details of the proofs of the main text. It summarizes well-known facts about the quantum theory of angular momentum and that of SU(d) generators. Further details will be presented elsewhere [S1].

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Angular momentum operators. Next, we summarize the fundamental equations for angular momentum operators [S2]. For particle with spin-j we have

$$(j_x^2+j_y^2+j_z^2)=j(j+1)\mathbb{1}. \tag{S1}$$

Since the angular momentum operators have identical spectra, it follows from Eq. (S1) that we can write

$$\mathrm{Tr}(j_x^2)=\frac{1}{3}j(j+1)(2j+1). \tag{S2}$$

Based on Eq. (S2), we get the constant for the orthogonality relation

$$\mathrm{Tr}(j_k j_l)=\delta_{kl}\frac{1}{3}j(j+1)(2j+1). \tag{S3}$$

For the sum of the squares of expectation values we have

$$\sum_{k=x,y,z}\langle j_k\rangle^2\le j^2. \tag{S4}$$

For $j=\frac{1}{2}$, for all pure states the equality holds for Eq. (S4).

Finally,

$$\sum_{l=x,y,z}\langle(j_l\otimes\mathbb{1}+\mathbb{1}\otimes j_l)^2\rangle\le 2j(2j+1). \tag{S5}$$

Hence, using Eq. (S1) we obtain

$$2j(j+1)+2\sum_{l=x,y,z}\langle j_l\otimes j_l\rangle\le 2j(2j+1). \tag{S6}$$

Thus, we arrive at the inequality

$$\sum_{l=x,y,z}\langle j_l\otimes j_l\rangle\le j^2. \tag{S7}$$

Local orthogonal observables. Here we summarize the results of Ref. [S3] for Local Orthogonal Observables (LOOs, [S4]). For a system of dimension d, these are d^2 observables λ_k such that

$$\mathrm{Tr}(\lambda_k\lambda_l)=\delta_{kl}. \tag{S8}$$

For a quantum state ϱ, LOOs have the following properties

$$\sum_{k=1}^{d^2}(\lambda_k)^2 \;=\; d\mathbb{1}, \tag{S9}$$

$$\sum_{k=1}^{d^2}\langle\lambda_k\rangle^2 \;=\; \mathrm{Tr}(\varrho^2)\le 1. \tag{S10}$$

Moreover, based on Ref. [S5] we know that

$$\sum_{k=1}^{d^2}\lambda_k\otimes\lambda_k=F, \tag{S11}$$

where F is the flip operator exchanging two qudits.

SU(d) generators. Next, we will use the results known for local orthogonal observables for SU(d) generators. For a system of dimension d, there are d^2-1 traceless SU(d) generators g_k with the property

$$\mathrm{Tr}(g_k g_l)=2\delta_{kl}. \tag{S12}$$

Thus, from SU(d) generators g_k we can obtain LOOS using

$$\lambda_k=\frac{1}{\sqrt{2}}g_k \tag{S13}$$

for $k=1,2,...,d^2-1$, and $\lambda_{d^2}=\frac{1}{\sqrt{d}}\mathbb{1}$.

After a derivation similar to that of Ref. [S3], we arrive at

$$\sum_{k=1}^{d^2-1}(g_k)^2=2\frac{d^2-1}{d}\mathbb{1}, \tag{S14}$$

$$\sum_{k=1}^{d^2-1}\langle g_k\rangle^2=2\left(\mathrm{Tr}(\varrho^2)-\frac{1}{d}\right)\le 2\left(1-\frac{1}{d}\right), \tag{S15}$$

$$\sum_{k=1}^{d^2-1}g_k\otimes g_k=2\left(F-\frac{1}{d}\mathbb{1}\right). \tag{S16}$$

Based on Eq. (S16), for bipartite symmetric states we have

$$\langle \sum_{k=1}^{d^2-1} g_k \otimes g_k \rangle = 2\left(+1-\frac{1}{d}\right), \tag{S17}$$

while for antisymmetric states we have

$$\langle \sum_{k=1}^{d^2-1} g_k \otimes g_k \rangle = 2\left(-1-\frac{1}{d}\right). \tag{S18}$$

It is important to stress that the inequalities presented are valid for all SU(d) generators, not only for Gell-Mann matrices.

Equations for the collective operators based on SU(d) generators.

Here we present some fundamental relations for the collective operators G_k. First of all, the length of the vector $\vec{G} = \{\langle G_k \rangle\}_{k=1}^{d^2-1}$ is maximal for a state of the form $|\Psi\rangle^{\otimes N}$. This can be seen as for such states $\vec{G} = N\vec{g}$ where $\vec{g} = \{\langle g_k \rangle_\Psi\}_{k=1}^{d^2-1}$, and knowing that for pure states $|\vec{g}|$ is maximal.

For the sum of the squares of G_k we obtain

$$\begin{aligned} \sum_k (G_k)^2 &= \sum_k \sum_n (g_k^{(n)})^2 + \sum_k \sum_{n\neq m} g_k^{(m)} g_k^{(n)} \\ &= 2N\frac{d^2-1}{d}\mathbb{1} + \sum_{n\neq m} 2\left(F_{mn} - \frac{\mathbb{1}}{d}\right). \end{aligned} \tag{S19}$$

Here we used Eq. (S14) and Eq. (S16). Based on Eq. (S19) and using $\langle F_{mn} \rangle \geq -1$, we can write

$$\sum_k \langle (G_k)^2 \rangle \geq \frac{2N}{d}(d+1)(d-N). \tag{S20}$$

Note that the bound on the right-hand side of Eq. (S20) cannot be zero if $N < d$. For $N = d$, the sum $\sum_k \langle (G_k)^2 \rangle$ is zero for the totally antisymmetric state for which $\langle F_{mn} \rangle = -1$ for all m, n.

Next, we will show that

$$\sum_k \langle G_k^2 \rangle = 0 \quad \Leftrightarrow \quad \sum_k (\Delta G_k)^2 = 0. \tag{S21}$$

In order to prove that, one has to notice that $\sum_k (\Delta G_k)^2 = 0$ implies $\sum_k (\Delta G'_k)^2 = 0$ for any set of SU(d) generators G'_k [S6]. This also implies $(\Delta B)^2 = 0$ for all traceless observables B. For every traceless D one can find traceless B_1 and B_2 such that $[B_1, B_2] = iD$ [S7] and hence $(\Delta B_1)^2 + (\Delta B_2)^2 \geq |\langle D \rangle|$. Hence, $\sum_k \langle (G_k)^2 \rangle = 0$ implies $\langle D \rangle = 0$ for all traceless observables D [S1].

As a consequence of Eq. (S20) and Eq. (S21), for $N < d$ we have $\sum_k (\Delta G_k)^2 > 0$. Hence, for d-dimensional systems states with less than d particles cannot have $\sum_k (\Delta G_k)^2 = 0$.

Moreover, for symmetric states we have $\langle F_{mn} \rangle = +1$ for all m, n, and based on Eq. (S19) we obtain

$$\sum_k \langle (G_k)^2 \rangle = \frac{2N}{d}(d-1)(d+N), \tag{S22}$$

which is the maximal value for $\sum_k \langle (G_k)^2 \rangle$. Similarly, for symmetric states,

$$\sum_k \langle (\tilde{G}_k)^2 \rangle = \sum_k \langle (G_k)^2 \rangle - \langle \sum_k \sum_n (g_k^{(n)})^2 \rangle \tag{S23}$$

is also maximal.

Naturally, these statements are also true for the angular momentum operators for the $j = \frac{1}{2}$ case, as these operators, apart from a constant factor, are SU(2) generators.

On the other hand, for the angular momentum operators for $j > \frac{1}{2}$ these statements are not true. In particular, $\langle \sum_k (J_k)^2 \rangle$ is not maximal for every symmetric state.

References

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[S6] Note that $\sum_k (\Delta G_k)^2 = \mathrm{Tr}(\gamma)$, where the covariance matrix is defined as $\gamma_{kl} = \frac{1}{2}(\langle \Delta G_k \Delta G_l \rangle + \langle \Delta G_l \Delta G_k \rangle)$. $\mathrm{Tr}(\gamma)$ is independent of the particular choice of the G_k matrices.

[S7] This is true because the group generated by G_k is a simple group. See also L. O'Raifeartaigh, *Group structure of gauge theories* (Cambridge University Press, New York, 1986).