Reference-frame-independent quantum metrology

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(Received 25 November 2024; accepted 2 March 2026; published 26 March 2026)

How can we perform a metrological task if only limited control over a quantum system is given? Here, we present systematic methods for conducting nonlinear quantum metrology in scenarios lacking a common reference frame. Our approach involves preparing multiple copies of quantum systems and then performing local measurements with randomized observables. First, we derive the metrological precision using an error-propagation formula based solely on local unitary invariants, which are independent of the chosen basis. Next, we provide analytical expressions for the precision scaling in various examples of nonlinear metrology involving two-body interactions, like the one-axis twisting Hamiltonian. Finally, we analyze our results in the context of local decoherence and discuss its influences on the observed scaling.

DOI: 10.1103/988w-6tpf

I. INTRODUCTION

Quantum metrology involves three stages: (1) preparing an initial probe state, (2) encoding a parameter θ through a state transformation, and (3) measuring the transformed state to extract information about θ. Each stage can be quantum or classical, including choices such as entangled or separable initial states, entangling or nonentangling state transformations, and measurements with multipartite operators or single-particle operators [1–7]. The precision of parameter estimation, denoted as $(\Delta\theta)^2$, naturally depends on the chosen approach.

A central goal in *quantum* metrology is to overcome the precision reachable in the classical regime. In a fully classical setup, the best achievable precision is known as the shot-noise limit [2,4–6] $(\Delta\theta)^2 \propto N^{-1}$, where N is the number of particles in a probe system. On the other hand, the presence of initial entanglement in the preparation stage enables scaling beyond the shot-noise limit, ultimately reaching the quadratic limit: $(\Delta\theta)^2 \propto N^{-2}$ [2,4–6]. Furthermore, even without initial entanglement, entangling transformations can yield a scaling significantly better than the quadratic scaling, expressed as $(\Delta\theta)^2 \propto N^{-2k+1}$ for integer values of k [8–17], and an exponential scaling proposed by Roy and Braunstein [18]. Note that the notion of the Heisenberg limit was originally introduced in Ref. [19], but later it was reformulated based on a detailed accounting of relevant resources [20,21].

For better accuracy, it is essential to have precise control of state preparation and parameter encoding, along with favorable measurements. In practice, however, uncontrollable experimental noise, such as magnetic field fluctuations for trapped ions or polarization rotations in optical fibers, can cause information loss in the encoding directions and disturb the calibration and alignment of the measurement settings. Such noise effects may result in the absence of a common Cartesian reference frame [associated with the SO(3) or SU(2) group]. Also, one might consider the possibility of adversarial attacks by malicious parties that could covertly disrupt the target system, perhaps nullifying quantum advantages and preventing higher precision.

What options are available in such a nonideal and black-box scenario where quantum control for the metrological tasks is limited? One approach is multiparameter metrology [22–32] to estimate the desired parameter as well as the error parameters in one go, but this may not work effectively if numerous unexpected variables change the system. Another approach is to establish a common reference frame, but this is known to be a resource-intensive process [33,34].

II. A NO-GO THEOREM

More specifically, let us regard a reference frame as an abstract coordinate system that allows for transforming unspeakable information (such as spin rotation) into speakable

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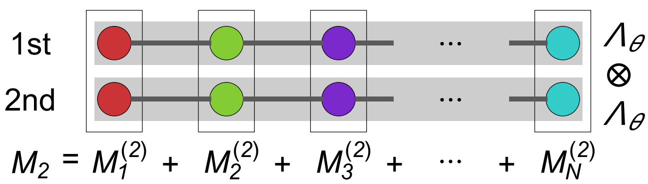

FIG. 1. Sketch of the quantum metrology scheme presented in this paper for two copies of an N-particle state with randomized measurements. The parameter θ is encoded in the first and second copies via $\Lambda_\theta \otimes \Lambda_\theta$ highlighted by gray color. Subsequently, a randomized measurement $M_2 = \sum_{i=1}^N M_i^{(2)}$ is performed on the two-copy state, where each local observable $M_i^{(2)}$ acts on the first and second copies (highlighted by a vertical box). This paper introduces systematic methods for evaluating the precision, denoted as $(\Delta\theta)^2$, in a reference-frame-independent manner.

information (such as classical bits represented as 0 or 1); for more details, see Ref. [34]. This is a general concept of how gyroscopes and clocks give a consistent way to measure orientation and time. Here, we consider reference frames in quantum systems that can be converted to another system with a different reference frame through a transformation involving an element denoted as $U \in \mathrm{SU}(d)$. For $d = 2$, the Cartesian frame in SO(3) is represented by the group SU(2), corresponding to spin rotations.

In fact, given an N-particle state, the absence of a shared reference frame between the particles excludes a quantum advantage in standard schemes of metrology. More precisely, we have the following statement.

No-go theorem. In the absence of a common reference frame (i.e., N parties do not share a reference frame), quantum metrology using an encoding of a parameter with the local unitary $V_L = e^{-i\theta H_L}$ for $H_L = \sum_{i=1}^N H_i$ is impossible. This holds even if multiple copies of the N-particle state are used.

This can be seen as follows: If each of the N parties has no prior knowledge about the local reference frame (related to local unitary $U_L = U_1 \otimes \cdots \otimes U_N$), then k copies of the state can be represented by

$$\mathcal{G}_k(\varrho) = \int dU_L\, (U_L \varrho U_L^\dagger)^{\otimes k}, \tag{1}$$

where the integral is taken over the Haar measure. Hence, $\mathcal{G}_k(\varrho)$ is invariant under any local unitaries, that is, $[\mathcal{G}_k(\varrho), V_L^{\otimes k}] = 0$ for any $V_L = e^{-i\theta H_L}$. Then, one cannot encode any information about the parameter θ from $V_L = e^{-i\theta H_L}$, making linear metrology impossible.

Consequently, it is natural to ask whether *nonlinear* metrology with nonlocal (entangling) unitary transformations is still possible even without a common reference frame. Here, we introduce metrological strategies in a reference-frame-independent manner. Our main idea is to prepare multiple copies of a quantum system and employ locally randomized measurement observables after a parameter encoding transformation, illustrated in Fig. 1. This strategy is inspired by previous works on randomized measurements used to characterize quantum correlations [35,36]. Although related scenarios were discussed in Refs. [37–42], randomized measurements were, to our knowledge, not employed in those studies.

In this paper, we present systematic methods to analyze the precision in quantum metrology without a common reference frame. In particular, we provide analytical expressions for the precision with several copies of the state using the error-propagation formula. Our formulation relies on local unitary invariants, which can be used for nonlinear quantum metrology rather than linear one. Then, we calculate the scaling behavior of the precision for various examples and discuss how decoherence effects can worsen metrological performance.

III. PARAMETER ESTIMATION AND ERROR PROPAGATION

Consider a quantum state of N d-dimensional particles (qudits) denoted as ϱ defined in the Hilbert space $\mathcal{H}_d^{\otimes N}$. Suppose that the initial state ϱ can be transformed by a quantum transformation Λ_θ to encode a parameter θ: $\varrho \to \varrho_\theta = \Lambda_\theta(\varrho)$. In the following, we focus on a unitary parameter encoding $V_\theta = e^{-i\theta H}$ with a (usually nonlocal) Hamiltonian H as the generator of the dynamics.

The parameter θ can be estimated from a measurement observable M. The estimation precision can be characterized by the error-propagation formula [5–7,43,44]

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M\rangle|^2}, \tag{2}$$

where $(\Delta M)^2 \equiv \langle M^2\rangle - \langle M\rangle^2$ and $\langle M\rangle$ is the expectation of M. Note that the precision $(\Delta\theta)^2$ depends on θ and that the above formula allows to study the situation where θ is small.

IV. MAIN METHOD

Let us consider k copies of the quantum state and perform a measurement on the system

$$\langle M_k\rangle = \mathrm{tr}\big(\varrho_\theta^{\otimes k} M_k\big), \tag{3}$$

where M_k acts on the k copies of the state defined in $(\mathcal{H}_d^{\otimes N})^{\otimes k}$. Here, we assume that the measurement observable M_k is *local* with respect to the parties,

$$M_k = \sum_{i=1}^N M_i^{(k)}, \tag{4}$$

where each $M_i^{(k)}$ acts (potentially nonlocally) on the k copies of the ith system (see also Fig. 1). As a choice of the observable $M_i^{(k)}$, we introduce a *randomized* (twirled) measurement observable

$$\Phi_k(\mathcal{O}) = \int dU\, (U^\dagger \mathcal{O} U)^{\otimes k}, \tag{5}$$

where $\mathcal{O}$ is a Hermitian operator defined in $\mathcal{H}_d$ and the integral is taken as the Haar random unitaries. By definition, this does not change under any unitary: $\Phi_k(V^\dagger \mathcal{O} V) = \Phi_k(\mathcal{O})$ for any unitary V.

For the sake of simplicity, we hereafter assume that $\mathcal{O}$ is traceless, $\mathrm{tr}(\mathcal{O}) = 0$, and normalized, $\mathrm{tr}(\mathcal{O}^2) = d$. We can then formulate the following:

Observation 1. Consider the two copies of an N-qudit system with $k = 2$ and $M_i^{(2)} = \Phi_2(\mathcal{O}_i)$. Then, the error-propagation formula leads to

$$(\Delta\theta)_2^2 = \frac{(d^2-1)N - 2S_1(\theta) + 2S_2(\theta) - S_1^2(\theta)}{|\partial_\theta S_1(\theta)|^2}, \quad (6)$$

where $S_1(\theta) = \sum_{i=1}^N [d\mathrm{tr}(\varrho_i^2) - 1]$ for the single-particle reduced state $\varrho_i = \mathrm{tr}_{\bar{i}}(\varrho_\theta)$ and $S_2(\theta) = \sum_{i<j}\{d^2\mathrm{tr}(\varrho_{ij}^2) - 1 - [d\mathrm{tr}(\varrho_i^2) - 1] - [d\mathrm{tr}(\varrho_j^2) - 1]\}$ for the two-particle reduced state $\varrho_{ij} = \mathrm{tr}_{\overline{ij}}(\varrho_\theta)$, where all particles up to i, j are traced out.

Proof. To derive Eq. (6), we substitute the observable in Eq. (4) with $M_i^{(2)} = \Phi_2(\mathcal{O}_i)$ into Eq. (2). For that, we first need to evaluate the Haar integral. In fact, assuming that $\mathcal{O}_i$ is traceless and normalized, one has

$$\Phi_2(\mathcal{O}_i) = \frac{1}{d^2-1}\big(d\mathbb{S}_i - \mathbb{1}_d^{\otimes 2}\big), \quad (7)$$

where $\mathbb{S}_i$ is the swap operator acting on both the first and second copies of the ith system: $\mathbb{S}\,|x\rangle\,|y\rangle = |y\rangle\,|x\rangle$; for details, see Refs. [45–48]. Using the property $\mathbb{S}_i^2 = \mathbb{1}_d$ and the swap trick $\mathrm{tr}[(X\otimes Y)\mathbb{S}] = \mathrm{tr}[XY]$ for operators X and Y, a straightforward calculation yields $\langle M_2\rangle = S_1(\theta)/(d^2-1)$. Similarly, we can arrive at Eq. (6). ■

We have five remarks on Observation 1. First, the sketch of this metrological scheme is illustrated in Fig. 1. Second, in general, the quantities S_l for integer $l \in [1, N]$ are known as the l-body sector length [49–52], which can be associated with the purity of the l-particle reduced states, leading to the identity $\sum_{l=1}^N S_l(\varrho) = d^N\mathrm{tr}(\varrho^2) - 1$. An important property is their invariance under local unitary transformations: $S_l(V_1\otimes\cdots\otimes V_N\varrho V_1^\dagger\otimes\cdots\otimes V_N^\dagger) = S_l(\varrho)$ for any unitary V_i for $i \in [1, N]$.

Third, the precision $(\Delta\theta)_2^2$ in Eq. (6) may remind us of the conventional spin-squeezing parameters [53–58] in the sense that the denominator relies on the reduced single-particle states and the numerator depends on both the reduced single-particle and two-particle states. In contrast to spin-squeezing parameters, the precision $(\Delta\theta)_2^2$ does not change under any parameter encoding by local unitary $V_L = e^{-i\theta H_L}$ for a local Hamiltonian $H_L = \sum_{i=1}^N H_i$, which agrees with the no-go theorem discussed above. On the other hand, it can be changed under some unitary $V_G = e^{-i\theta H_G}$ for a nonlocal interaction Hamiltonian H_G.

Finally, one might generalize our approach to further multicopy scenarios or other observables, perhaps resulting in additional local unitary invariants with higher degrees. However, it would be demanding to find the simple expression by the analytical evaluation of Haar integrals, and it may not necessarily lead to higher precision.

In the following, we consider four copies to derive the precision achievable with higher-order quantities. For the sake of simplicity, we focus on qubits ($d = 2$) where we can, without loss of generality, take $\mathcal{O} = \sigma_z$. Then, we have the following:

Observation 2. Consider the four copies of an N-qubit system, that is, $k = 4$, $d = 2$, and $M_i^{(4)} = \Phi_4(\sigma_z^{(i)})$. Then, the error-propagation formula leads to

$$(\Delta\theta)_4^2 = \frac{15N - 20S_1(\theta) + 8F_1(\theta) + 2F_2(\theta) - 3F_1^2(\theta)}{3|\partial_\theta F_1(\theta)|^2}, \quad (8)$$

where $F_1(\theta) = \sum_{i=1}^N [2\mathrm{tr}(\varrho_i^2) - 1]^2$ for $\varrho_i = \mathrm{tr}_{\bar{i}}(\varrho_\theta)$ and $F_2(\theta) = \sum_{i<j}\{[\mathrm{tr}(T_{ij}T_{ij}^\top)]^2 + 2\mathrm{tr}(T_{ij}T_{ij}^\top T_{ij}T_{ij}^\top)\}$ with the matrix T_{ij} with the elements $[T_{ij}]_{\mu\nu} = \mathrm{tr}(\varrho_{ij}\sigma_\mu\otimes\sigma_\nu)$ for $\varrho_{ij} = \mathrm{tr}_{\overline{ij}}(\varrho_\theta)$ and $\mu, \nu = x, y, z$.

The proof of Observation 2 is given in Appendix A. We remark that F_1 and F_2 are also invariants under local unitaries, where F_2 for $N = 2$ is known as one of the Makhlin invariants [59]. Note also that the precision $(\Delta\theta)_4^2$ depends on the one- and two-body correlations of ϱ only, which can be traced back to the fact that M_4 is a one-body observable, and $(\Delta\theta)_4^2$ contains the variance of it.

Here, it should be essential to notice that $(\Delta\theta)_2^2$ in Eq. (6) results from the two copies of N particles, while $(\Delta\theta)_4^2$ in Eq. (8) results from the four copies of N particles. To compare both correctly, let us introduce the gain relative to the shot-noise limit

$$G_k = \frac{1}{kN(\Delta\theta)_k^2}. \quad (9)$$

Note that $G_k > 1$ implies higher precision beyond the shot-noise limit.

V. NONLINEAR HAMILTONIAN DYNAMICS

Here, we show several scalings in the proximity of $\theta = 0$ based on our results. For that, we consider the estimation precision in the limit of $\theta \to 0$. We present the following result:

Observation 3. Consider that $|\psi_\theta\rangle = e^{-i\theta H}\,|1\rangle^{\otimes N}$ and $H = J_x^2$, where $J_x = \frac{1}{2}\sum_{i=1}^N \sigma_x^{(i)}$. Then, the gain in Eq. (9) is obtained as

$$\lim_{\theta\to 0} G_2 = \frac{N-1}{4}, \quad (10a)$$

$$\lim_{\theta\to 0} G_4 = \frac{3(N-1)}{8}, \quad (10b)$$

for $k = 2$ and $k = 4$, respectively.

Proof. To prove this Observation, we need to compute all the terms $S_1(\theta)$, $S_2(\theta)$, $F_1(\theta)$, and $F_2(\theta)$. Since the state $|\psi_\theta\rangle$ is symmetric under exchange for any two qubits, it is sufficient to focus on one of the reduced two-qubit states and multiply its results by some factors like N or $N(N-1)/2$ in the end. With the help of the result in Ref. [60], we can immediately find their explicit expressions; for details, see Appendix B. ■

We remark that the nonlinear dynamics with $H = J_x^2$ is called the one-axis twisting Hamiltonian. This nonlinear dynamics is known to produce spin-squeezing entanglement [6,53,58] and also many-body Bell correlations [61]. In Appendix D, we will consider another Hamiltonian dynamics and show that the gain G_k in Eq. (9) for $k = 2, 4$ scales exponentially, similarly to the example of Roy and Braunstein in Ref. [18]. In general, quantum entanglement generated by the unitary dynamics is necessary to achieve high values of G_k.

Note that our scaling $G_k \propto N$ in Eqs. (10a) and (10b), that is, $(\Delta\theta)^2 \propto 1/N^2$, was found in Ref. [62] for product states with separable measurements in the one-axis twisting Hamiltonian (moreover, the better precision $(\Delta\theta)^2 \propto 1/N^3$ was discussed [12–14]). In these works, however, a shared

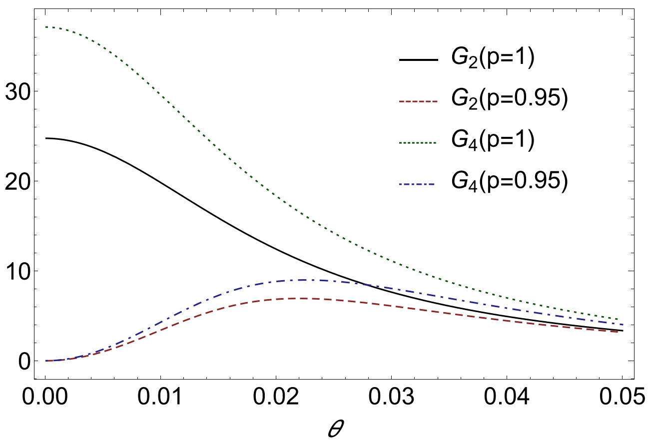

FIG. 2. Sensitivity of the metrological gain defined in Eq. (9) to parameter shifts based on Observation 3 in $N = 100$, where p denotes the noise parameter in the local depolarizing channel in Eq. (11).

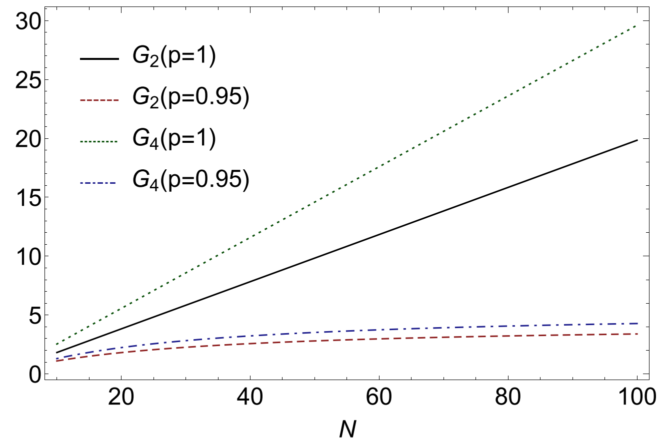

FIG. 3. Growth in the metrological gain defined in Eq. (9) for an increasing number of particles based on Observation 3 with a fixed $\theta = 1/N$, where p denotes the noise parameter in the local depolarizing channel in Eq. (11).

reference frame between the particles was assumed, and our results show that is not needed.

VI. DECOHERENCE

Here, we analyze how decoherence can influence the metrological gain. As a typical decoherence model, we consider the so-called depolarizing noise channel for a quantum state $\sigma \in \mathcal{H}_d$ (see Ref. [63]):

$$\mathcal{E}_p(\sigma) = p\sigma + \frac{1-p}{d}\mathbb{1}_d, \tag{11}$$

where $0 \leqslant p \leqslant 1$. In the following, let us discuss a scenario that a pure initial state is transformed by nonlinear unitary dynamics, and then each particle is *locally* affected by the depolarizing channel with the same local error parameter $\Lambda_\theta(\varrho) = \mathcal{E}_p^{\otimes N}(V_\theta \varrho V_\theta^\dagger)$. In this scenario, the precision can be obtained from the formulas given for the noiseless case in Eqs. (6) and(8) by replacing $S_l(\theta)$ and $F_l(\theta)$ for $l = 1, 2$ by

$$S_l(\theta) \to p^{2l} S_l(\theta), \tag{12a}$$

$$F_l(\theta) \to p^{4l} F_l(\theta). \tag{12b}$$

In Figs. 2 and 3, we plot the gains $G_k(p)$ for a noise parameter p of Observation 3 and compare the noiseless and a noisy case with $p = 0.95$. In Fig. 2, we can find that the maximal gain in the noisy case cannot be achieved by the limit of $\theta \to 0$ unlike the noiseless case, and the optimal value of θ can be shifted depending on p and N [64,65]. In Fig. 3, we illustrate the growth in both gains for increasing particles for fixed $\theta = 1/N$. We remark that the metrological gain decreases due to noise effects, where the precision in Eqs. (10a) and (10b) is diminished.

VII. COLLECTIVE RANDOMIZATION

So far, we have discussed the reference-frame-independent metrological scheme using locally independent randomized observables based on Eq. (5). As shown in Fig. 1, we have considered individual randomizations for each local measurement on the two-copy space, as $M_2 = \sum_{i=1}^N \Phi_2(\sigma_z^{(i)})$ in Eq. (4). Here, we will introduce a *collective* randomization scheme that uses multilateral simultaneous rotations on all subsystems, motivated by the recent work in Ref. [66].

More precisely, instead of M_2, we consider a collective randomized observable on the two-copy system:

$$\mathcal{X}_2 = \int dU\, (U^{\dagger \otimes N} J_z U^{\otimes N})^{\otimes 2}, \tag{13}$$

where $J_z = \frac{1}{2}\sum_{i=1}^N \sigma_z^{(i)}$ is known as the collective angular momentum acting on the N-qubit system. Here, U is a local unitary on a single particle and the difference to the scheme before is that the same U is applied to all parties. Then, we can formulate the final result:

Observation 4. Consider the two copies of an N-qubit system with the collective randomized observable $\mathcal{X}_2$. The error-propagation formula then leads to

$$(\Delta\theta)^2_{C_2} = \frac{f(N) + B(\theta) - [S_1(\theta) + K_1(\theta)]^2}{|\partial_\theta[S_1(\theta) + K_1(\theta)]|^2}, \tag{14}$$

where $f(N) = 3N(-2N + 3)$ and

$$B(\theta) = 2[-S_1(\theta) - 2K_1(\theta) + S_2(\theta) + K_2(\theta)] + K_2'(\theta) + 16(N-1)\sum_{\mu=1}^{3}\langle J_\mu^2\rangle. \tag{15}$$

Here, we define that $K_1(\theta) = \sum_{i\neq j}^N \sum_{\mu=1}^3 \operatorname{tr}[(\varrho_i \otimes \varrho_j)(\sigma_\mu \otimes \sigma_\mu)]$ for $\varrho_i = \operatorname{tr}_{\bar{i}}(\varrho_\theta)$, $K_2(\theta) = \sum_{i\neq j\neq k} \operatorname{tr}(T_{ij} T_{ik}^\top)$, and $K_2'(\theta) = \sum_{i\neq j\neq k\neq l} \operatorname{tr}(T_{ik} T_{jl}^\top)$ with the matrix T_{ij} with the elements $[T_{ij}]_{\mu\nu} = \operatorname{tr}(\varrho_{ij}\sigma_\mu \otimes \sigma_\nu)$ for $\varrho_{ij} = \operatorname{tr}_{\overline{ij}}(\varrho_\theta)$.

The proof of Observation 4 is given in Appendix C. In order to distinguish with the precision $(\Delta\theta)_2^2$ in Observation 1, we add the subscript C_2 in the precision $(\Delta\theta)^2_{C_2}$. By definition, $(\Delta\theta)^2_{C_2}$ does not change under any *collective* local unitary $V^{\otimes N}$.

We apply this result to nonlinear metrology with the one-axis twisting Hamiltonian $H = J_x^2$. From a similar calculation with Observation 3 (see Appendix B), the gain

$G_{C_2} = [2N(\Delta\theta)^2_{C_2}]^{-1}$ is obtained as

$$\lim_{\theta\to 0} G_{C_2} = \frac{N^2(N-1)}{2N(3N-5)+8}. \tag{16}$$

This scaling is the same as in Eq. (10a).

VIII. EXPERIMENTAL IMPLICATIONS

The one-axis twisting Hamiltonian $H = J_x^2$ can be realized in several physical systems. In trapped cold atoms, even addressing the particles is possible [67,68]. In a Bose-Einstein condensate, this can be realized by collisional interactions between atoms [55]. Also, the generation of Haar random unitaries can be implemented by stochastic quantum walks on the integrated photonic chips [69,70] or a polarization controller operating in scrambling mode [71]. It is also possible to realize such operations in parallel in an ensemble of many multiqudit quantum systems [72].

We note that the Haar integral on $\Phi_k(\mathcal{O})$ as in Eq. (5) can be implementable by measuring permutation operators, such as Eq. (7) for $k=2$. More generally, the Schur-Weyl duality implies that any observable $\Phi_k(\mathcal{O})$ can be expressed as linear combination of permutation operators [48], i.e., $\Phi_k(\mathcal{O}) = \sum_{\pi\in\mathrm{Sym}_k} c_\pi W_\pi$, where c_π is a coefficient that depends on $\mathcal{O}$ and W_π is a permutation operator corresponding to a permutation π in the symmetric group Sym_k of order k.

IX. CONCLUSION

We have proposed (collective) reference-frame-independent metrological schemes with two and four copies of a quantum state. Our formulation is invariant under (collective) local unitaries, which enables us to perform nonlinear metrology with nonlocal transformations. We analytically computed the precision for several relevant cases, showing that they go beyond the shot-noise limit.

There are several research directions in which our work can be generalized. First, it would be interesting to show analytical examples of various types of scaling with the nonlinear Hamiltonian $H = J_x^k$ [10,11,15]. Second, by the spirit of spin squeezing, finding metrologically meaningful uncertainty relations from random measurements may give fundamental limitations of precision. Finally, our method may encourage the further development of parameter estimation tasks in terms of multicopy metrology [73], multiparameter scenarios [27,29], or temperature estimation in quantum thermodynamics [74].

ACKNOWLEDGMENTS

We would like to thank Francesco Albarelli, Iagoba Apellaniz, Jan Lennart Bönsel, Stefan Nimmrichter, Luca Pezzé, Augusto Smerzi, Róbert Trényi, Nikolai Wyderka, and Benjamin Yadin for discussions. This work was supported by the DAAD, the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation, projects no. 447948357, No. 440958198, and No. 56343716), the Sino-German Center for Research Promotion (Project No. M-0294), the ERC (Consolidator Grant No. 683107/TempoQ), the German Federal Ministry of Research, Technology and Space (Project QuKuK, Grant No. 16KIS1618K and Project BeRyQC, Grant No. 13N17292), National Research, Development and Innovation Office of Hungary (NKFIH) (Grant No. 2019-2.1.7-ERA-NET-2021-00036, Advanced Grant No. 152794), and the National Research, Development and Innovation Office of Hungary within the Quantum Information National Laboratory of Hungary. We acknowledge the support of the EU (QuantERA MENTA, QuantERA QuSiED), the Spanish MCIU (No. PCI2022-132947), and the Basque Government (No. IT1470-22). We acknowledge the support of Grant No. PID2021-126273NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe." We thank the "Frontline" Research Excellence Programme of the NKFIH (Grant No. KKP133827). G.T. acknowledges a Bessel Research Award of the Humboldt Foundation. We acknowledge Project No. TKP2021-NVA-04, which has been implemented with the support provided by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Office, financed under the TKP2021-NVA funding scheme. We thank support from Horizon Europe programme HORIZON-CL4-2022-QUANTUM-02-SGA via Project No. 101113690 (PASQuanS2.1) and JST ASPIRE (JPMJAP2339).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PROOF OF OBSERVATION 2

Observation 2. Consider the four copies of an N-qubit system, that is, $k=4$, $d=2$, and $M_i^{(4)} = \Phi_4(\sigma_z^{(i)})$. Then, the error-propagation formula leads to

$$(\Delta\theta)_4^2 = \frac{15N - 20S_1(\theta) + 8F_1(\theta) + 2F_2(\theta) - 3F_1^2(\theta)}{3|\partial_\theta F_1(\theta)|^2}, \tag{A1}$$

where $F_1(\theta) = \sum_{i=1}^N [2\mathrm{tr}(\varrho_i^2) - 1]^2$ for $\varrho_i = \mathrm{tr}_{\bar{i}}(\varrho_\theta)$ and

$$F_2(\theta) = \sum_{i<j} \left\{ \left[\mathrm{tr}\left(T_{ij}T_{ij}^\top\right)\right]^2 + 2\mathrm{tr}\left(T_{ij}T_{ij}^\top T_{ij}T_{ij}^\top\right)\right\}, \tag{A2}$$

with the matrix T_{ij} with the elements $[T_{ij}]_{\mu\nu} = \mathrm{tr}(\varrho_{ij}\sigma_\mu \otimes \sigma_\nu)$ for $\varrho_{ij} = \mathrm{tr}_{\overline{ij}}(\varrho_\theta)$ and $\mu, \nu = x, y, z$.

Proof. We begin by evaluating the form of $\langle M_4\rangle$ as follows:

$$\begin{aligned}\langle M_4\rangle &= \sum_{i=1}^N \mathrm{tr}\left[\varrho_i^{\otimes 4}\Phi_4\left(\sigma_z^{(i)}\right)\right]\\ &= \frac{1}{2^4}\sum_{i=1}^N \sum_{a,b,c,d} r_a^{(i)} r_b^{(i)} r_c^{(i)} r_d^{(i)} \mathcal{I}(a,b,c,d)\\ &= \frac{1}{5}F_1(\theta).\end{aligned} \tag{A3}$$

Here, in the first line, we use that $\Phi_4(\sigma_z^{(i)})$ only acts on the four copies of the ith system. In the second line, we denote that for $r_\mu^{(i)} = \mathrm{tr}[\varrho_i\sigma_\mu^{(i)}]$ for $\mu = a, b, c, d = x, y, z$ and

$$\mathcal{I}(a,b,c,d) = \int dU\, Z_{U,a}^{(i)} Z_{U,b}^{(i)} Z_{U,c}^{(i)} Z_{U,d}^{(i)}, \tag{A4}$$

where $Z_{U,a}^{(i)} = \mathrm{tr}[\sigma_a^{(i)} U^\dagger \sigma_z^{(i)} U]$. In the third line, we apply the following formula:

$$\mathcal{I}(a,b,c,d) = \frac{16}{15}(\delta_{a,b}\delta_{c,d} + \delta_{a,c}\delta_{b,d} + \delta_{a,d}\delta_{b,c}), \tag{A5}$$

given in Ref. [71] and introduce the fourth-order quantity

$$F_1(\theta) = \sum_{i=1}^N r_i^4, \tag{A6}$$

where $r_i^2 = \sum_{\mu=x,y,z}[r_\mu^{(i)}]^2 = 2\mathrm{tr}[\varrho_i^2] - 1$.

Next, we will evaluate the expression of the variance: $(\Delta M_4)^2 = \langle M_4^2\rangle - \langle M_4\rangle^2$, where

$$\langle M_4^2\rangle = \sum_i \langle \Phi_4(\sigma_z^{(i)})^2\rangle + \sum_{i\neq j}\langle \Phi_4(\sigma_z^{(i)})\Phi_4(\sigma_z^{(j)})\rangle. \tag{A7}$$

The first term in $\langle M_4^2\rangle$ is given by

$$\begin{aligned}
&\sum_i \langle \Phi_4(\sigma_z^{(i)})^2\rangle \\
&= \sum_i \mathrm{tr}\{[\varrho_i^{\otimes 4}\Phi_4(\sigma_z^{(i)})]_X \otimes [\Phi_4(\sigma_z^{(i)})]_Y\, \mathbb{S}_{X,Y}\} \\
&= \frac{1}{2^4}\sum_i \int dU_X \int dU_Y [\mathrm{tr}(\chi_x^{(i)} \otimes \upsilon_y^{(i)} \mathbb{S}_{x,y})]^4 \\
&= \frac{1}{2^4\times 2^4}\sum_i \int dU_X \int dU_Y \Big[\sum_\alpha \mathcal{Z}_{U_X,\alpha}^{(i)} Z_{U_Y,\alpha}^{(i)}\Big]^4 \\
&= \frac{1}{2^4\times 2^4}\sum_i \sum_{\alpha,\beta,\gamma,\delta} \mathcal{I}(\alpha,\beta,\gamma,\delta)\sum_{j=0}^4 \mathcal{C}_j^{(i)} \\
&= \frac{1}{5\times 15}[15N - 20S_1(\theta) + 8F_1(\theta)].
\end{aligned} \tag{A8}$$

Here, in the first equality, we divide the squared term into two different spaces X, Y inversely using the swap trick mentioned in the proof of Observation 1: $\mathrm{tr}(XY^2) = \mathrm{tr}[(XY\otimes Y)\mathbb{S}]$. The swap $\mathbb{S}_{X,Y}$ acts on the eight-qubit system, where each system $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ is the four-copy of a single-qubit system.

In the second equality, we use that the swap operator in many qubits can be realized by the swap operators in individual qubits [75], that is,

$$\mathbb{S}_{X,Y} = \mathbb{S}_{x_1,y_1}\otimes \mathbb{S}_{x_2,y_2}\otimes \mathbb{S}_{x_3,y_3}\otimes \mathbb{S}_{x_4,y_4}. \tag{A9}$$

Also, we denote that

$$\chi_x^{(i)} = U_X^\dagger \sigma_z^{(i)} U_X + \sum_{a=x,y,z} r_a^{(i)}\sigma_a^{(i)} U_X^\dagger \sigma_z^{(i)} U_X, \tag{A10a}$$

$$\upsilon_y^{(i)} = U_Y^\dagger \sigma_z^{(i)} U_Y, \tag{A10b}$$

where $r_a^{(i)} = \mathrm{tr}[\varrho_i \sigma_a^{(i)}]$.

In the third equality, we apply the formulas

$$\mathbb{S} = \frac{1}{2}\Big(\mathbb{1}_2^{\otimes 2} + \sum_{\alpha=x,y,z}\sigma_\alpha\otimes\sigma_\alpha\Big), \tag{A11a}$$

$$\sigma_p\sigma_q = \delta_{p,q}\mathbb{1}_2 + i\sum_{r=x,y,z}\varepsilon_{p,q,r}\sigma_r, \tag{A11b}$$

with the Kronecker-delta symbol $\delta_{p,q}$ and the Levi-Civita symbol $\varepsilon_{p,q,r}$, and denote that

$$\mathcal{Z}_{U_X,\alpha}^{(i)} = Z_{U_X,\alpha}^{(i)} + i\sum_{a,k=x,y,z}\varepsilon_{\alpha,a,k} r_a^{(i)} Z_{U_X,k}^{(i)}, \tag{A12}$$

where $Z_{U_X,\alpha}^{(i)} = \mathrm{tr}[\sigma_\alpha^{(i)} U_X^\dagger \sigma_z^{(i)} U_X]$.

In the fourth equality, we denote that

$$\int dU_X\, \mathcal{Z}_{U_X,\alpha}^{(i)}\mathcal{Z}_{U_X,\beta}^{(i)}\mathcal{Z}_{U_X,\gamma}^{(i)}\mathcal{Z}_{U_X,\delta}^{(i)} = \sum_{j=0}^4 \mathcal{C}_j^{(i)}, \tag{A13}$$

where $\alpha,\beta,\gamma,\delta = x,y,z$ and the label j in $\mathcal{C}_j^{(i)}$ represents the number of times the imaginary unit i is multiplied.

In the final equality, we indeed evaluate all the terms in $\mathcal{C}_j^{(i)}$ and simplify the expression. Note that $\mathcal{C}_1^{(i)} = \mathcal{C}_3^{(i)} = 0$ for any i due to the properties of the Kronecker delta and Levi-Civita symbol.

Let us continue the computation of the variance. The second term in $\langle M_4^2\rangle$ can be given by

$$\begin{aligned}
&\sum_{i\neq j}\langle \Phi_4(\sigma_z^{(i)})\Phi_4(\sigma_z^{(j)})\rangle \\
&= \sum_{i\neq j}\mathrm{tr}[\varrho_{ij}^{\otimes 4}\Phi_4(\sigma_z^{(i)})\Phi_4(\sigma_z^{(j)})] \\
&= \frac{1}{4^4}\sum_{i\neq j}\sum_{\mathbf{a},\mathbf{b}} t_{a_1b_1}^{(ij)} t_{a_2b_2}^{(ij)} t_{a_3b_3}^{(ij)} t_{a_4b_4}^{(ij)}\mathcal{I}(\mathbf{a})\mathcal{I}(\mathbf{b}) \\
&= \frac{2}{5\times 15}F_2(\theta).
\end{aligned} \tag{A14}$$

In the second equality, we denote that $\mathbf{a} = (a_1, a_2, a_3, a_4)$, $\mathbf{b} = (b_1, b_2, b_3, b_4)$, and $t_{a_pb_p}^{(ij)} = \mathrm{tr}(\varrho_{ij}\sigma_{a_p}\otimes\sigma_{b_p})$ for $a_p, b_p = x, y, z$. In the third equality, we use that the sector length can be given by $S_2(\theta) = \sum_{i<j}\mathrm{tr}(T_{ij}T_{ij}^\top) = \sum_{i<j}[4\mathrm{tr}(\varrho_{ij}^2) - 1 - S_1(\theta)]$ with the matrix $[T_{ij}]_{ab} = t_{ab}^{(ij)}$ and introduce the fourth-order two-body quantity

$$F_2(\theta) = \sum_{i<j}\{[\mathrm{tr}(T_{ij}T_{ij}^\top)]^2 + 2\mathrm{tr}(T_{ij}T_{ij}^\top T_{ij}T_{ij}^\top)\}. \tag{A15}$$

Hence, we can complete the proof. ■

APPENDIX B: DERIVATION OF OBSERVATION 3

Observation 3. Consider that $|\psi_\theta\rangle = e^{-i\theta H}|1\rangle^{\otimes N}$ and $H = J_x^2$, where $J_x = \frac{1}{2}\sum_{i=1}^N \sigma_x^{(i)}$. Then, the gain in Eq. (9) is obtained as

$$\lim_{\theta\to 0} G_2 = \frac{N-1}{4}, \tag{B1a}$$

$$\lim_{\theta\to 0} G_4 = \frac{3(N-1)}{8}, \tag{B1b}$$

for $k = 2$ and $k = 4$, respectively.

Here, we will give the the explicit expressions of $S_1(\theta), S_2(\theta), F_1(\theta)$, and $F_2(\theta)$. Let us begin by recalling that all the single-qubit and two-qubit reduced states ϱ_i and ϱ_{ij} are the same for $i, j = 1, \dots, N$. Then, we denote that $\varrho_i = \varrho_1$ and $\varrho_{ij} = \varrho_2$. According to the result in Ref. [60], for $r_\mu = \mathrm{tr}(\varrho_1 \sigma_\mu)$ and $t_{\mu\nu} = \mathrm{tr}(\varrho_2 \sigma_\mu \otimes \sigma_\nu)$, we can have

$$r_x = r_y = 0, \quad r_z = -\cos^{N-1}(\theta), \tag{B2a}$$

$$t_{xx} = t_{xz} = t_{zx} = t_{yz} = t_{zy} = 0, \tag{B2b}$$

$$t_{yy} = \tfrac{1}{2}[1 - \cos^{N-2}(2\theta)], \tag{B2c}$$

$$t_{zz} = \tfrac{1}{2}[\cos^{N-2}(2\theta) + 1], \tag{B2d}$$

$$t_{xy} = t_{yx} = \sin(\theta)\cos^{N-2}(\theta). \tag{B2e}$$

This yields

$$S_1(\theta) = N \sum_{\mu=x,y,z} r_\mu^2 = N\cos^{2N-2}(\theta), \tag{B3a}$$

$$\begin{aligned} S_2(\theta) &= \frac{N(N-1)}{2} \sum_{\mu,\nu=x,y,z} t_{\mu\nu}^2 \\ &= \frac{N(N-1)}{4}[\cos^{2(N-2)}(2\theta) \\ &\quad + 4\sin^2(\theta)\cos^{2N-4}(\theta) + 1], \end{aligned} \tag{B3b}$$

$$F_1(\theta) = N\Big[\sum_{\mu=x,y,z} r_\mu^2\Big]^2 = N\cos^{4N-4}(\theta), \tag{B3c}$$

$$\begin{aligned} F_2(\theta) &= \frac{N(N-1)}{2}\Bigg\{\Big[\sum_{\mu,\nu=x,y,z} t_{\mu\nu}^2\Big]^2 \\ &\quad + 2\sum_{\mu.\nu,\xi,\kappa=x,y,z} t_{\mu\nu}t_{\mu\kappa}t_{\xi\nu}t_{\xi\kappa}\Bigg\} \\ &= \frac{N(N-1)}{4}\{\cos^{4(N-2)}(2\theta) \\ &\quad - 8\sin^2(\theta)\cos^{2N-4}(\theta)\cos^{N-2}(2\theta) \\ &\quad + \cos^{2(N-2)}(2\theta)[8\sin^2(\theta)\cos^{2N-4}(\theta) + 4] \\ &\quad + (4\sin^2(\theta)\cos^{2N-4}(\theta) + 1)^2\}. \end{aligned} \tag{B3d}$$

Substituting these expressions into the precisions $(\Delta\theta)_2^2$ in Eq. (6) and $(\Delta\theta)_4^2$ in Eq. (8) and taking the limit $\theta \to 0$, we can arrive at the results in Eqs. (B1a) and (B1b).

APPENDIX C: PROOF OF OBSERVATION 4

Observation 4. Consider the two copies of an N-qubit system with the collective randomized observable $\mathcal{X}_2$. The error-propagation formula leads to

$$(\Delta\theta)_{C_2}^2 = \frac{f(N) + B(\theta) - [S_1(\theta) + K_1(\theta)]^2}{|\partial_\theta[S_1(\theta) + K_1(\theta)]|^2}, \tag{C1}$$

where $f(N) = 3N(-2N+3)$ and

$$\begin{aligned} B(\theta) &= 2[-S_1(\theta) - 2K_1(\theta) + S_2(\theta) + K_2(\theta)] + K_2'(\theta) \\ &\quad + 16(N-1)\sum_{\mu=1}^{3}\langle J_\mu^2\rangle. \end{aligned} \tag{C2}$$

Here, we define that $K_1(\theta) = \sum_{i\neq j}^N \sum_{\mu=1}^3 \mathrm{tr}[(\varrho_i \otimes \varrho_j)(\sigma_\mu \otimes \sigma_\mu)]$ for $\varrho_i = \mathrm{tr}_{\bar{i}}(\varrho_\theta)$ and

$$K_2(\theta) = \sum_{i\neq j\neq k} \mathrm{tr}\big(T_{ij}T_{ik}^\top\big), \quad K_2'(\theta) = \sum_{i\neq j\neq k\neq l} \mathrm{tr}\big(T_{ik}T_{jl}^\top\big), \tag{C3}$$

with the matrix T_{ij} with the elements $[T_{ij}]_{\mu\nu} = \mathrm{tr}(\varrho_{ij}\sigma_\mu \otimes \sigma_\nu)$ for $\varrho_{ij} = \mathrm{tr}_{\overline{ij}}(\varrho_\theta)$ and $\mu, \nu = x, y, z$.

Proof. We begin by denoting

$$\begin{aligned} \mathcal{X}_2 &= \int dU\, (U^{\dagger\otimes N} J_z U^{\otimes N})^{\otimes 2} \\ &= \frac{1}{4}\sum_{i,j=1}^N \int dU\, \big(U^\dagger \sigma_z^{(i)} U\big) \otimes \big(U^\dagger \sigma_z^{(j)} U\big) \\ &\equiv \frac{1}{4}\sum_{i,j=1}^N \Phi_{ij}, \end{aligned} \tag{C4}$$

where

$$\Phi_{ij} = \frac{1}{3}(2\mathbb{S}_{ij} - \mathbb{1}) = \frac{1}{3}\sum_{\mu=x,y,z} \sigma_\mu^{(i)} \otimes \sigma_\mu^{(j)}. \tag{C5}$$

Here, i denotes the ith system in the first copy and j denotes the jth system in the second copy. Then, we can immediately evaluate the form of $\langle \mathcal{X}_2 \rangle$ as follows:

$$\begin{aligned} \langle \mathcal{X}_2 \rangle = \mathrm{tr}(\varrho^{\otimes 2}\mathcal{X}_2) &= \frac{1}{4}\sum_{i,j=1}^N \mathrm{tr}[(\varrho_i \otimes \varrho_j)\Phi_{ij}] \\ &= \frac{1}{3\times 4}[S_1(\theta) + K_1(\theta)], \end{aligned} \tag{C6}$$

where $\varrho_i = \mathrm{tr}_{\bar{i}}(\varrho_\theta)$ is the single-particle reduced state.

Next, we will evaluate the form of the variance: $(\Delta\mathcal{X}_2)^2 = \langle \mathcal{X}_2^2 \rangle - \langle \mathcal{X}_2 \rangle^2$. Here, a straightforward calculation yields

$$\begin{aligned} \mathcal{X}_2^2 = \frac{1}{16}\Bigg\{ &\sum_i \Phi_{ii}^2 + \sum_{i\neq j}\big[\Phi_{ii}\Phi_{jj} + 4\Phi_{ii}\Phi_{ij} + 2\Phi_{ij}^2\big] \\ &+ \sum_{i\neq j\neq k}\big[4\Phi_{ij}\Phi_{ik} + 2\Phi_{ii}\Phi_{jk}\big] + \sum_{i\neq j\neq k\neq l}\Phi_{ij}\Phi_{kl}\Bigg\}, \end{aligned} \tag{C7}$$

where we used that $\Phi_{ij} = \Phi_{ji}$. To find the explicit form of $(\Delta\mathcal{X}_2)^2$, we must evaluate all these expectations. A long calculation leaves us with

$$\sum_i \langle \Phi_{ii}^2 \rangle = \tfrac{1}{9}[3N - 2S_1(\theta)], \tag{C8a}$$

$$\sum_{i\neq j} \langle \Phi_{ii}\Phi_{jj} \rangle = \tfrac{2}{9}S_2(\theta), \tag{C8b}$$

$$\sum_{i\neq j} \langle \Phi_{ii}\Phi_{ij} \rangle = \frac{1}{9}\Bigg[4\sum_{\mu=x,y,z} \mathrm{tr}\big(\varrho_\theta J_\mu^2\big) - 3N\Bigg], \tag{C8c}$$

$$\sum_{i\neq j} \langle \Phi_{ij}^2 \rangle = \tfrac{1}{9}[3N(N-1) - 2K_1(\theta)], \tag{C8d}$$

$$\sum_{i\neq j\neq k} \langle \Phi_{ij}\Phi_{ik} \rangle = \frac{(N-2)}{9}\Bigg[4\sum_{\mu=x,y,z} \mathrm{tr}\big(\varrho_\theta J_\mu^2\big) - 3N\Bigg], \tag{C8e}$$

$$\sum_{i\neq j\neq k} \langle \Phi_{ii}\Phi_{jk}\rangle = \frac{1}{9}\sum_{i\neq j\neq k} K_2(\theta), \tag{C8f}$$

$$\sum_{i\neq j\neq k\neq l} \langle \Phi_{ij}\Phi_{kl}\rangle = \frac{1}{9}\sum_{i\neq j\neq k} K_2'(\theta). \tag{C8g}$$

Summarizing these terms, we can complete the proof. Here, it might be useful for some readers to note that

$$\Phi_{ij}^2 = \frac{1}{9}\sum_{\mu,\nu=x,y,z} \sigma_\mu^{(i)}\sigma_\nu^{(i)} \otimes \sigma_\mu^{(j)}\sigma_\nu^{(j)} = \frac{1}{9}(5\mathbb{1} - 4\mathbb{S}_{ij}), \tag{C9}$$

where we recall Eq. (C5) and use the property in Eq. (A11b) and $\sum_{\mu,\nu=x,y,z}\varepsilon_{\mu,\nu,\xi}\varepsilon_{\mu,\nu,\xi'} = 2\delta_{\xi,\xi'}$. ■

Remark. Consider a permutationally invariant state: $\varrho = \mathbb{S}_{ij}\varrho\mathbb{S}_{ij}$ for all $i, j \in \{1, 2, \ldots, N\}$ with $i \neq j$; for details, see Ref. [76]. Since $\mathbb{S}(X\otimes Y)\mathbb{S} = Y \otimes X$ for operators X, Y, any permutationally invariant state ϱ has the same Bloch vector $\boldsymbol{r} = \boldsymbol{r}_i$ and the same matrix $T = T_{ij}$ for all $i \neq j$ with $[T]_{\mu\nu} = [T]_{\nu\mu}$. Thus, we can simplify the expressions

$$S_1(\theta) + K_1(\theta) = N^2 r^2(\theta), \tag{C10a}$$

$$\begin{aligned} B(\theta) = N\{&r^2(2-4N) + (N-1) \\ &\times [(3+N(N-3))t_2(\theta) \\ &+ 4(N-1)t_1(\theta) + 12]\}, \end{aligned} \tag{C10b}$$

where we denoted that $r^2(\theta) = |\boldsymbol{r}|^2$, $t_1(\theta) = \mathrm{tr}(T)$, and $t_2(\theta) = \mathrm{tr}(T^2)$. To find the above simplification, we used

$$S_1 = Nr^2, \tag{C11a}$$

$$K_1 = N(N-1)r^2, \tag{C11b}$$

$$S_2 = \frac{N(N-1)}{2}t_2, \tag{C11c}$$

$$K_2 = N(N-1)(N-2)t_2, \tag{C11d}$$

$$K_2' = N(N-1)(N-2)(N-3)t_2, \tag{C11e}$$

$$\sum_{\mu=x,y,z}\langle J_\mu^2\rangle = \frac{1}{4}[3N + N(N-1)t_1], \tag{C11f}$$

where we abbreviated the notation of θ.

APPENDIX D: EXPONENTIAL SCALING

Here, we formulate the exponential scaling:

Observation 5. Consider that $|\psi_\theta^n\rangle = e^{-i\theta H}|0\rangle^{\otimes N}$ and $H = Q_n$ for $n = 1, 2$ such that $Q_1 + iQ_2 = (\sigma_x + i\sigma_y)^{\otimes N}$. Then, the gain in Eq. (9) is obtained as

$$\lim_{\theta\to 0} G_2 = \frac{4^N}{N+1}, \tag{D1a}$$

$$\lim_{\theta\to 0} G_4 = \frac{3\times 2^{2N+1}}{3N+1}, \tag{D1b}$$

for $k = 2$ and $k = 4$, respectively.

Remark. The Hamiltonian Q_n for $n = 1, 2$ is the same as the Hermitian operator used in Mermin-type inequalities [77–80]. This Hamiltonian model was considered by Roy and Braunstein in Ref. [18].

Proof. According to Ref. [18], we have that

$$|\psi_\theta^1\rangle = \cos(\theta')|0\rangle^{\otimes N} - i\sin(\theta')|1\rangle^{\otimes N}, \tag{D2a}$$

$$|\psi_\theta^2\rangle = \cos(\theta')|0\rangle^{\otimes N} + \sin(\theta')|1\rangle^{\otimes N}, \tag{D2b}$$

where $\theta' = 2^{N-1}\theta$. To proceed, we need to evaluate all the terms $S_1(\theta)$, $S_2(\theta)$, $F_1(\theta)$, and $F_2(\theta)$. As a more general case, let us consider the N-qubit pure asymmetric Greenberger–Horne–Zeilinger (GHZ) state:

$$|\mathrm{GHZ}_{\alpha,\beta}\rangle = \alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}, \tag{D3}$$

where $|\alpha|^2 + |\beta|^2 = 1$ for complex coefficients α, β. The reduced two-qubit state in any systems $i, j = 1, \ldots, N$ is given by

$$\begin{aligned} \varrho_{ij}^{(2)} &= \mathrm{tr}_{\overline{ij}}(|\mathrm{GHZ}_{\alpha,\beta}\rangle\langle \mathrm{GHZ}_{\alpha,\beta}|) \\ &= \frac{1}{4}\left\{\mathbb{1}_2^{\otimes 4} + \Delta\left[\sigma_z^{(i)} + \sigma_z^{(j)}\right] + \sigma_z^{(j)}\otimes\sigma_z^{(i)}\right\}, \end{aligned} \tag{D4}$$

with $\Delta = |\alpha|^2 - |\beta|^2$. Thus, we can immediately find

$$S_1(\mathrm{GHZ}_{\alpha,\beta}) = N\Delta^2, \tag{D5a}$$

$$S_2(\mathrm{GHZ}_{\alpha,\beta}) = \frac{N(N-1)}{2}, \tag{D5b}$$

$$F_1(\mathrm{GHZ}_{\alpha,\beta}) = N\Delta^4, \tag{D5c}$$

$$F_2(\mathrm{GHZ}_{\alpha,\beta}) = \frac{3N(N-1)}{2}. \tag{D5d}$$

Substituting these into the form in Eq. (9) and taking the limit $\theta \to 0$, we can complete the proof. ■

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