Quantum Chromodynamics meets Quantum Information

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Outline

1. **Motivation**
   - What can be interesting for people working on QCD in Quantum Information?
2. **Quantum information science**
3. **Quantum entanglement**
   - Pure states: is it a pair or is it not a pair?
   - Mixed states: is it a pair or is it not a pair?
   - Local Operations and Classical Communication (LOCC)
4. **Examples for entanglement in QCD**
   - Quarks and gluons
   - Entanglement criterion for $d = 3$-dimensional particles
   - Detection of singlets
5. **Quantum optical systems and QCD**
   - Cold gases on a lattice

What can be interesting for QCD people in Quantum Information?

- Entanglement theory can help to recognize real two- and three-particle states.
- QCD-like systems can be realized with cold atoms.

Quantum Information Science

- Quantum optics 60's (collective manipulation of particles)
  - matter-light interaction, laser, etc.

- Quantum information 80's/90's- (individual manipulation of particles)
  - Few-body systems
    - cold trapped ions
    - cold atoms on an optical lattice
    - photons
  - Many-body systems
    - cold atomic ensembles
    - Bose-Einstein Condensates of cold atoms

Quantum Information Science II

- Quantum information 80's/90's- (continued)
  - Entanglement theory
  - Quantum computers and algorithms for quantum computers (prime factoring)
  - Quantum cryptography, quantum communication

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Pure states: is it a pair or is it not a pair?

Separability

A bipartite pure state is separable if and only if it is a product state. Otherwise the state is called entangled.

- Easy to check. The reduced state of the second party is obtained as

$$\varrho_{2\mathrm{red}} = \mathrm{Tr}_1(|\Psi_{12}\rangle\langle\Psi_{12}|)$$

- If $|\Psi_{12}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$ if and only iff

$$\mathrm{Tr}(\varrho_{2\mathrm{red}}^2) = 1.$$

- Alternatively: ... if and only if

$$S(\varrho_{2\mathrm{red}}) = 0,$$

where S is the von Neumann entropy

Pure states: is it a pair or is it not a pair? II

Von Neumann entropy of a block measures

- the purity of the block,
- that is, entanglement with the neighborhood.

Pure states: is it a pair or is it not a pair? III

- Example 1: product state

$$|\Psi_{12}\rangle = \frac{1}{2}(|0\rangle+|1\rangle)(|0\rangle+|1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

A particle is "independent" from the other. The single particle reduced state is pure.

$$\varrho_{2\mathrm{red}} = \frac{1}{2}(|0\rangle+|1\rangle)(\langle 0|+\langle 1|).$$

- Example 2: entangled state

$$|\Psi_{12}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The single particle reduced state is completely mixed.

$$\varrho_{2\mathrm{red}} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

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Mixed states: is it a pair or is it not a pair?

Separability

A quantum state is called separable if it can be written as [Werner, 1989]

$$\varrho = \sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)},$$

where p_k form a probability distribution ($p_k > 0, \sum_k p_k = 1$), and $\varrho_n^{(k)}$ are single-particle density matrices. A state that is not separable is called entangled.

- The purity or the von Neumann entropy cannot detect entanglement any more so easily.

Mixed states: is it a pair or is it not a pair?

Entanglement of formation

For mixed states the entanglement of formation is given as a convex roof

$$E_F = \min_{|\Psi_k\rangle, p_k} \sum_k p_k S(\mathrm{Tr}_1(|\Psi_k\rangle\langle\Psi_k|)),$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- In general, there is no closed formula.
- Easy to compute for small systems or for systems with some symmetry.

Mixed states: is it a pair or is it not a pair? III

Example

Let us mix

$$|\Psi_{12}^{(1)}\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle)$$

and

$$|\Psi_{12}^{(2)}\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle - |11\rangle),$$

as

$$\varrho = \frac{1}{2}\left(|\Psi_{12}^{(1)}\rangle\langle\Psi_{12}^{(1)}| + |\Psi_{12}^{(2)}\rangle\langle\Psi_{12}^{(2)}|\right).$$

- Question: Is this entangled? It is a mixture of entangled states.
- Answer: no, since ϱ can be written as

$$\varrho = \frac{1}{2}\,(|00\rangle\langle 00| + |11\rangle\langle 11|)\,.$$

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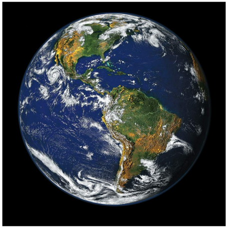
Local Operations and Classical Communication (LOCC)

- LOCC are
  - local unitaries
    $$U_1 \otimes U_2$$
  - local von Neumann (or POVM) measurements
    $$M \otimes Identity$$
  - local unitaries or measurements conditioned on measurement outcomes on the other party.
- LOCC cannot create entangled states from a separable state.

Local Operations and Classical Communication (LOCC)



PARTICLE_1

PARTICLE_2

No entanglement can be created without real two-body quantum dynamics.

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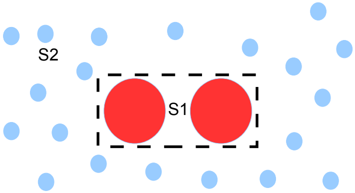
Entanglement and QCD

- A quark/antiquark pair in a gluon environment and look at the entropy of the color state of the quarks.
[Buividovich, Kuvshinov, AIP Conf. Proc. 1205, 26 (2010).]

- The color state is a singlet
$\Rightarrow$ purity is 1 and the entropy is zero.



Entanglement and QCD II

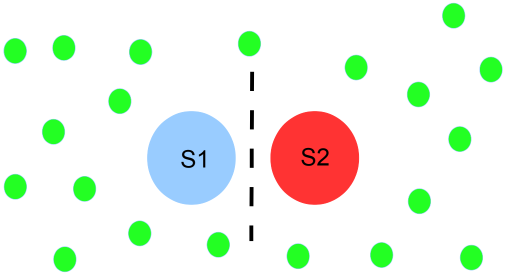
- The entanglement between the quarks and the gluons tells us only indirect information about the entanglement between the quarks.

- Monogamy of entanglement (official terminology!):
  - when the two quarks are maximally entangled (=singlet), they cannot be entangled with the environment.

[V. Coffman et al., Phys. Rev. A 61, 052306 (2000);
B. M. Terhal, Linear Algebra Appl. 323, 61 (2001).]

Entanglement and QCD III

- Question: Is there entanglement between the two quarks?
- Answer: More complicated question. The color state is mixed, thus the entanglement cannot be so easily computed.



Entanglement between the quarks

- If the state is not entangled then there are no pairs, $E_F = 0$!
- If the state is entangled, then there are pairs. For two-body singlets $E_F =$maximal!

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Criterion to exclude separability

- Entanglement measures are hard to compute. Let us look for some sufficient condition for entanglement.

- g_l with $l = 1, 2, ..., 8$ are the Gell-Mann matrices.

- Collective operators:

$$G_l := g_l^{(1)} - (g_l^{(2)})^*.$$

.

- We also need the variances

$$(\Delta G_l)^2 := \langle G_l^2 \rangle - \langle G_l \rangle^2.$$

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Criterion to exclude separability II

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1)$$

with $d = 3$ and $N = 2$.

- Any state that violates this is entangled.
- For two-body color singlets, the LHS=0!

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth, Optimal spin squeezing inequalities for arbitrary spin, Phys. Rev. Lett. (2011).]

Similar ideas for N>2 for tree-body singlets

- g_l with $l = 1, 2, ..., 8$ are the Gell-Mann matrices.
- Collective operators:

$$G_l := \sum_{k=1}^{N} g_l^{(k)},$$

Criterion for three-particle entanglement (trion

A condition for states without three-particle entanglement is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-2)$$

with $d = 3$ and $N = 3$.

- Any state that violates this is three-particle entangled.
- Recognizes three-particle color singlets! For the singlet the LHS=0.

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth,
Optimal spin squeezing inequalities for arbitrary spin,
Phys. Rev. Lett. (2011).]

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Cold gas experiments and QCD

Quantum Matter

PERSPECTIVE

Quantum Gases

Immanuel Bloch

Ultracold quantum gases are proving to be a powerful model system for strongly interacting electronic many-body systems. This Perspective explores how such atomic ensembles can help to unravel some of the outstanding open questions in the field.

When matter is cooled down close to zero temperature, particles can interact in a cooperative way and form novel states of matter with striking properties—superconductors, superfluids, or fractional quantum Hall liquids. Similar phenomena can now be observed in a dilute gas of atoms, five to six orders of magnitude less dense than the air surrounding us. Here, degenerate bosonic and fermionic quantum gases trapped in magnetic or optical traps are generated at temperatures in the nanokelvin regime (*1*). Whereas initial research concentrated on weakly interacting quantum states [for example, on elucidating the coherent matter wave features of Bose-Einstein condensates (BECs) and their superfluid properties], research has now turned toward strongly interacting bosonic and fermionic systems (*2*, *3*). In these systems, the interactions between the particles dominate over their kinetic energy, making them difficult to tackle theoretically but also opening the path to novel ground states with collective properties of the many-body system. This has given rise to the hope of using the highly controllable quantum gases as model systems for condensed-matter physics, along the lines of a quantum simulator, as originally suggested by Feynman (*4*).

Two prominent examples have dominated the research in this respect: (i) the transition from a superfluid to a Mott insulator of bosonic atoms

Feshbach resonances. Such bosonic composites can themselves undergo Bose-Einstein condensation, thus fundamentally altering the properties of the many-body system. When a true two-body bound state exists between the particles, the composite bosonic particle is simply a molecule, albeit very large, whereas in the case of attractive interactions without a two-body bound state the composite pair can be seen to be related to a BCS-type Cooper pair, which can then undergo condensation. It is the possibility of changing almost all the underlying param-

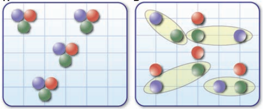

Fig. 1. Three-species fermionic atoms (red, green, and blue spheres) in an optical lattice can form two distinct phases when the interactions between the atoms are tuned. In the first case of strong attractive interactions between the atoms, they join as "trions" (**A**), whereas in the second case of weaker interactions, a color superfluid is formed (**B**), in which atoms pair up between only two species. The two phases have strong analogies to the baryonic phase (A) and the color

observe exotic forms of superconductivity such as the Fulde-Ferrell-Larkin-Ovchinnikov superconducting phase (*13*, *14*), where particles condense into pairs with nonzero momentum. Early experiments have produced degenerate mixtures of two fermionic atomic species (*15*) and two fermionic species with an additional third bosonic component (*16*), and both are progressing quickly toward exploiting Feshbach resonances to control the interactions between the fermionic atoms.

For lattice-based systems, efforts are under way to explore the feasibility of using ultracold atoms as quantum simulators for strongly interacting many-body systems. For example, in the famous class of high-Tc superconductors, such as the CuO compounds, one observes that these form antiferromagnetically ordered ground states when undoped. Upon doping, and thereby changing the effective filling in the system, the antiferromagnetic order is destroyed and a superconducting phase with *d*-wave symmetry of the order parameter emerges (*17*) (Fig. 2). What exactly happens during the transition and how it can be described theoretically is currently a subject of heated debates and one of the fundamental unsolved problems in the field of condensed-matter physics. Cold-atom researchers are currently trying to determine whether they can help to resolve some of these issues (*18*). As a starting point, several groups are preparing to observe antiferromagnetically ordered states in two-component Fermi mixtures in an optical lattice. To achieve this, however, one needs to cool the many-body system to challenging temperatures

Hubbard model with three-state particles

$$\hat{H} = -t\sum_{\langle i,j\rangle,\alpha} \hat{c}^{+}_{i\alpha}\hat{c}_{j\alpha} + \sum_{\alpha\neq\beta}\sum_{i}\frac{U_{\alpha\beta}}{2}(\hat{n}_{i\alpha}\hat{n}_{i\beta}), \qquad (1)$$

with $\hat{c}^{+}_{i\alpha}$ the creation operator of a fermionic atom of color $\alpha=1,2,3$ at site i, and $\hat{n}_{i\alpha}=\hat{c}^{+}_{i\alpha}\hat{c}_{i\alpha}$. In the tunneling term, $\langle i,j\rangle$ implies the restriction to nearest neighbor sites, and the tunneling matrix element is approximately given by $t=E_R(2/\sqrt{\pi})s^{3/4}e^{-2s^{1/2}}$, where $E_R=\frac{\hbar^2q^2}{2m}$ is the recoil energy, q is the wave vector of the lasers, m is the mass of the atoms, $s=V_0/E_R$, and V_0 is the depth of the periodic potential.[9,10] We neglect the effects of the confining potential in Eq. (1), which would correspond to a site-dependent potential term in the Hamiltonian. The interaction strength $U_{\alpha\beta}$ between colors α and β is related to the corresponding s-wave scattering length, $a_{\alpha\beta}$, as $U_{\alpha\beta}=E_R a_{\alpha\beta} q\sqrt{8/\pi}s^{3/4}$.[9,10] Note that fermions with identical colors do not interact with each other.

For the sake of simplicity, we shall first consider the attractive case with $U_{\alpha\beta}=U<0$. This case could be realized by loading the ^{6}Li atoms into an optical trap in a large magnetic field, where the scattering lengths become large and negative, $a_{\alpha\beta}\approx a_s\approx -2500a_0$, for all three scattering channels, 12, 13, and 23.[20]

Introducing the usual Gell–Mann matrices, $\lambda^{a}_{\alpha\beta}$ ($a=1,\ldots,8$), it is easy to see that global SU(3) transformations $\exp(i\Sigma_i\Sigma_{a\alpha\beta}\phi_a\hat{c}^{+}_{i\alpha}\lambda^{a}_{\alpha\beta}\hat{c}_{i\beta})$ commute with the Hamiltonian, which thus also conserves the total number of fermions for each color, $\hat{N}_\alpha=\Sigma_i\hat{n}_{i\alpha}$. This conservation of particles is only approximate because in reality the number of the atoms in the trap continuously decreases due to different scattering processes. Here, however, we shall neglect this slow loss of atoms and keep the density ρ_α of atoms for color α as well as the overall filling factor $\rho\equiv\frac{1}{3}\Sigma_\alpha\rho_\alpha$ fixed.

Let us first focus on the case of equal densities, $\rho_\alpha=\rho$. For small attractive $U<0$, the ground state is a color superfluid:[15] atoms from two of the colors form the Cooper pairs and an s-wave superfluid, while the third color remains an unpaired Fermi liquid. However, as we discussed in Ref. 16, for large attractive interactions, this superfluid state becomes unstable, and instead of Cooper pairs, it is more likely to form three-atom bound states, the so-called "trions." These trions are color singlet fermions, and for large $|U|$

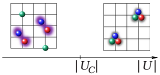

FIG. 2. (Color online) The ground states for $|U|\ll|U_C|$ and $|U|\gg|U_C|$ can be calculated by perturbation theory. The former is a BCS state which breaks the SU(3) symmetry, and the latter is a trionic state with three-particle singlet bound states.

In order to get analytic expressions, we shall study the ground state in $d=\infty$ dimensions. Then, to reach a meaningful limit and to get finite kinetic energy, one has to scale the hopping as $t=\frac{t^*}{2\sqrt{d}}$, with t^* fixed. In this limit, however, trions become immobile. Therefore, the $d\to\infty$ trionic states are well approximated as

$$|T_\Lambda\rangle = \prod_{i\in\Lambda}\hat{c}^{+}_{i1}\hat{c}^{+}_{i2}\hat{c}^{+}_{i3}|0\rangle, \qquad (3)$$

where Λ denotes a subset of sites where trions sit. We can calculate the energy of this state in infinite dimensions: a single trion has an energy $3U$, thus the energy of such a state per lattice site is given by $E_T/N=3U\rho$, with E_T the total energy of the system and N the number of lattice sites.

Clearly, the two ground states obtained by the perturbative expansions have different symmetries: the superfluid state breaks SU(3) invariance, while the trionic state does not. Therefore, there must be a phase transition between them. Note that, relying on symmetries only, this argument is very robust and carries over to *any* dimensions. In infinite dimensions, we find that trions are immobile. However, this is only an artifact of infinite dimensions and in finite dimensions, a superconductor-Fermi liquid phase transition should occur.

One could envision that some other order parameter also emerges and masks the phase transition discussed here. Preliminary results (not discussed here) suggest that indeed a charge density state forms at large values of $|U|$, but except for half-filling, which is a special case not discussed here, we do not see any other relevant order parameter that could in-

Literature

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- Á. Rapp, G. Zaránd, C. Honerkamp, and W. Hofstetter, Phys. Rev. Lett. 98, 160405 (2007).

Summary

- We discussed the possible connection between quantum chromodynamics and quantum information science
- In particular, we discussed entanglement theory.

For our criterion, see

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