Practical methods for witnessing genuine multi-qubit entanglement in the vicinity of symmetric states

Géza Tóth[1,2,3], Witlef Wieczorek[4,5], Roland Krischek[4,5],
Nikolai Kiesel[4,5,6], Patrick Michelberger[4,5], and Harald Weinfurter[4,5]

[1]Theoretical Physics, The University of the Basque Country, Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Research Institute for Solid State Physics and Optics, Budapest, Hungary
[4]Max-Planck-Institut für Quantenoptik, Garching, Germany
[5]Fakultät für Physik, Ludwig-Maximilians-Universität, Garching, Germany
[6]Institute for Quantum Optics and Quantum Information, Vienna, Austria

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Outline

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Motivation

- In many quantum experiments only local measurements are possible.
- We still would like to detect entanglement with few measurements.
- Entanglement detection schemes for cluster states and graph states are already available.

  GT and O. ühne, PRL and PRA 2005;
  N. Kiesel, C. Schmid, U. Weber, GT, O. Gühne, R. Ursin, and H. Weinfurter PRL 2005.

- We would like to develop methods for symmetric states, as such states appear often in many-qubit experiments.

  G. Tóth, J. Opt. Soc. Am. B 2007;
  N. Kiesel, C. Schmid, G. Tóth, E. Solano, and H. Weinfurter, PRL 2007.

Outline

Basic definitions I

- A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

  Here $|\Psi\rangle$ is an N-qubit state.

- A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

- If a state is not biseparable then it is called genuine multi-partite entangled.

Basic definitions II

- A state mixed with white noise is given as

$$\rho(p_{\text{noise}}) = (1 - p_{\text{noise}})\rho + p_{\text{noise}}\rho_{\text{noise}}$$

  where p_{noise} is the ratio of noise and ρ_{noise} is the noise. If we consider white noise then $\rho_{\text{noise}} = \mathbb{1}/2^N$.

- The noise tolerance of a witness $\mathcal{W}$ is characterized by the largest p_{noise} for which we still have $\mathrm{Tr}(\mathcal{W}\rho) < 0$.

Basic definitions III

- A single local measurement setting is the basic unit of experimental effort.
- A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.

$A^{(1)}$	$A^{(2)}$	$A^{(3)}$	$A^{(4)}$	$A^{(5)}$	...	$A^{(N)}$

- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)}\rangle, \langle A^{(1)}A^{(3)}\rangle, \langle A^{(1)}A^{(2)}A^{(3)}\rangle ...$$

Basic methods for designing witnesses

Three methods for designing witnesses:

- Projector witness, i.e., witness defined with the projector to a highly entangled quantum state
- Witness based on the projector witness
- Witness independent of the projector witness

Projector-based witness

- A witness detecting genuine multi-qubit entanglement in the vicinity of a pure state $|\Psi\rangle$ is

$$\mathscr{W}_{\Psi}^{(P)} := \lambda_{\Psi}^2 \mathbb{1} - |\Psi\rangle\langle\Psi|,$$

  where λ is the maximum of the Schmidt coefficients for $|\Psi\rangle$, when all bipartitions are considered.
  M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, PRL 2004.

- A symmetric witness operator can always be decomposed as

$$P = \sum c_k A_k \otimes A_k \otimes A_k \otimes ... \otimes A_k.$$

  G. Tóth, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, NJP 2009.

- For symmetric operators, the number of settings needed is increasing polynomially with the number of qubits.
  G. Tóth, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, NJP 2009.

Witnesses based on the projector witness

- We construct witnesses that are easier to measure than the projector witness.

- Idea: If $\mathscr{W}^{(P)}$ is the projector witness and

$$\mathscr{W} - \alpha\mathscr{W}^{(P)} \geq 0$$

  is fulfilled for some $\alpha > 0$, then $\mathscr{W}$ is also a witness.
  Similar idea for cluster states: GT and O. Gühne, PRL and PRA 2005.

- Hence, $\langle\mathscr{W}\rangle$ can be used to fidelity estimation.
  GT and O. Gühne, PRL and PRA 2005.

- We look for the optimal witness $\mathscr{W}$ with the largest tolerance to noise possible.

- This has been done analitically for cluster states. Now we use *semidefinite programming.*
  GT, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Wei*nfurter,* NJP 2009.

Witnesses independent from the projector witness

- Witnesses without any relation to the projector witness.
- With an easily measurable operator M, we make a witness of the form

$$\mathscr{W} := c\mathbb{1} - M,$$

where c is some constant.

- We have to set c to

$$c = \max_{|\Psi\rangle \in \mathscr{B}} \langle M \rangle_{|\Psi\rangle},$$

where $\mathscr{B}$ is the set of biseparable states.

Witnesses independent from the projector witness II

- Simple numerical optimization may not find the global maximum.
- We can look for the maximum for states that have a positive partial transpose

$$c' = \max_{I} \max_{\rho \geq 0, \rho^{T_I} \geq 0} \langle M \rangle_\rho,$$

  for which $c' \geq c$.
- This can be done with *semidefinite programing.*
  A.C. Doherty, P.Parrilo, and F.M. Spedalieri, PRA 2005.

Outline

Witnesses for the Dicke state: Projector

- The symmetric Dicke states are defined as

$$|D_N^{(m)}\rangle := \binom{N}{m}^{-\frac{1}{2}} \sum_k \mathscr{P}_k(|1_1, 1_2, ..., 1_m, 0_{m+1}, ..., 0_N)\rangle,$$

where the summation is over all different permutations.

- For the states considered in our work, projector-based witnesses are given by

$$\mathscr{W}^{(P)}_{\mathrm{D(N,N/2)}} := \frac{1}{2}\frac{N}{N-1}\mathbb{1} - |D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|,$$

G. Tóth, J. Opt. Soc. Am. B 2007.

$$\mathscr{W}^{(P)}_{\mathrm{D(N,1)}} := \frac{N-1}{N}\mathbb{I} - |D_N^{(1)}\rangle\langle D_N^{(1)}|.$$

Witnesses for the Dicke state: Projector II

- The decomposition of the projector is

$$
\begin{aligned}
64|D_6^{(3)}\rangle\langle D_6^{(3)}| = & -0.6[\mathbb{1}]+0.3[x\pm\mathbb{1}]-0.6[x]+0.3[y\pm\mathbb{1}]-0.6[y] \\
& +0.2[z\pm\mathbb{1}]-0.2[z]+0.2\mathrm{Mermin}_{0,z} \\
& +0.05[x\pm y\pm\mathbb{1}]-0.05[x\pm z\pm\mathbb{1}]-0.05[y\pm z\pm\mathbb{1}] \\
& -0.05[x\pm y\pm z]+0.2[x\pm z]+0.2[y\pm z]+0.1[x\pm y] \\
& +0.6\mathrm{Mermin}_{x,z}+0.6\mathrm{Mermin}_{y,z}.
\end{aligned}
$$

W. Wieczorek *et. al*, PRL 2009. Another decompostion: Prevedel *et al.*,PRL 2009.

- Notation: $[x+y]=(\sigma_x+\sigma_y)^{\otimes 6}$, $[x+y+\mathbb{1}]=(\sigma_x+\sigma_y+\mathbb{1})^{\otimes 6}$, etc.
- The $\pm$ sign denotes a summation over the two signs, i.e., $[x\pm y]=[x+y]+[x-y]$.
- 21 settings are needed to measure the projector based on this decomosition.

Witnesses for the Dicke state: Projector III

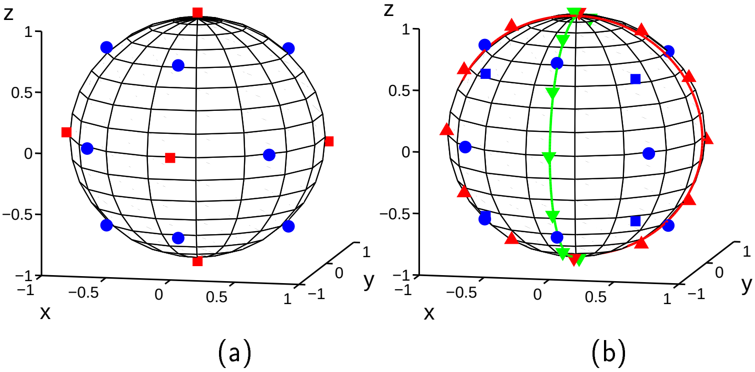

Figure: Measurement settings for projectors to the the (a) four-qubit and (b) six-qubit Dicke states

Witnesses for the Dicke state: Projector-based

- We look for the witness $\mathcal{W}$ with the largest noise tolerance that fulfills:

1. $\mathcal{W}$ is a linear combination of certain basis operators B_k, that is, $\mathcal{W} = \sum_k c_k B_k$,
2. $\mathcal{W} - \alpha \mathcal{W}^{(P)}_{\mathrm{D}(6,3)} \geq 0$ with some $\alpha > 0$. [GT and O. Gühne, PRL and PRA 2005]

- Optimal c_k's come from semidefinite programming.

- For the two-setting case, we set $\{B_k\} = \{\mathbb{1}, J_x^2, J_y^2, J_x^4, J_y^4, J_x^6, J_y^6\}$.

X	X	X	...	X

Y	Y	Y	...	Y

- For the three-setting case, we set $\{B_k\} = \{\mathbb{1}, J_x^2, J_y^2, J_z^2, J_x^4, J_y^4, J_z^4, J_x^6, J_y^6, J_z^6\}$.

X	X	X	...	X

Y	Y	Y	...	Y

Z	Z	Z	...	Z

Witnesses for the Dicke state: Independent

- For symmetric Dicke states, a general form a witness is

$$\mathscr{W}^{(I3)}_{\mathrm{D(N,m)}} := c_q - (J_x^2 + J_y^2) + q(J_z - \langle J_z \rangle_{|D_N^{(m)}\rangle})^2,$$

where c_q and q are constants.

Comparison of witnesses

	State	Witness	Number of settings	Noise tolerance
		Projector	21	0.4063
6-qubit	$D_6^{(3)}$	Projector-based	3	0.2735
		Projector-based	2	0.1391
		Independent	2	0.1091
5-qubit	$D_5^{(2)}$	Independent	2	0.1046
4-qubit	W	Projector	7	0.2667
		Independent	3	0.1476
	$D_4^{(2)}$	Projector	9	0.3556
		Projector-based	2	0.2759

Table: List of witnesses

Summary

- We discussed practical methods for constructing entanglement witnesses for systems up to 10 qubits.
- We applied our witnesses for an experiment aiming to observe a six-qubit symmetric Dicke state.
- For more details, see our papers:
- Witnesses:
  GT, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, New J. Phys. **11,** 083002 (2009)
- Experiment:
  W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, and H. Weinfurter, Phys. Rev. Lett. **103**, 020504 (2009).

Talk by W. WIECZOREK Q52.5 Thursday 15:00