Optimal generalized variance and quantum Fisher information

G. Tóth[1,2,3] and D. Petz[4,5]

[1]Theoretical Physics, University of the Basque Country UPV/EHU, Bilbao, Spain
[2]ikerbasque Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Center for Physics, Budapest, Hungary
[4]Rényi Instiute for Mathematics, Budapest, Hungary
[5]Department for Mathematical Analysis,
Budapest University of Technology and Economics,
Budapest, Hungary

DPG Meeting, Stuttgart, March 2012


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Hammerfest
Murmansk
Tromsø
Kiruna
White Sea
Arkhangel'sk
Reykjavík
ICELAND
Norwegian Sea
Arctic Circle
NORWAY
Luleå
Oulu
Umeå
FINLAND
Lake Onega
Trondheim
SWEDEN
Gulf of Bothnia
Tampere
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
Gävle
Turku
Helsinki
St. Petersburg
RUSSIA
Bergen
Oslo
Stockholm
ALAND ISLANDS
Gulf of Finland
Tallinn
ESTONIA
Rockall (U.K.)
ORKNEY ISLANDS
HEBRIDES
Stavanger
North Atlantic Ocean
Aberdeen
Göteborg
Gotland
Riga
LATVIA
Glasgow
Edinburgh
North Sea
DENMARK
Copenhagen
Malmö
Baltic Sea
Öland
LITHUANIA
Vilnius
Vitsyebsk
Smolensk
Belfast
UNITED KINGDOM
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
Leeds
Liverpool
Manchester
Kaliningrad
RUSSIA
Bornholm
Minsk
Mahilyow
BELARUS
Hrodna
Homyel'
Birmingham
Cardiff
London
Hamburg
Bremen
Berlin
Gdańsk
Poznań
POLAND
Warsaw
Łódź
Wrocław
Brest
Kiev
Amsterdam
NETH.
Rotterdam
Celtic Sea
Essen
Cologne
Leipzig
Brussels
BEL.
Lille
Bonn
GERMANY
Prague
CZECH REPUBLIC
Brno
Kraków
Rivne
UKRAINE
L'viv
English Channel
Guernsey (U.K.)
Jersey (U.K.)
Frankfurt am Main
Luxembourg
LUX.
Paris
Stuttgart
SLOVAKIA
Bratislava
Chernivtsi
Chişinău
MOLDOVA
Iaşi
Mykolayiv
Strasbourg
Munich
Vienna
AUSTRIA
Budapest
Cluj-Napoca
Nantes
LIECH.
HUNGARY
ROMANIA
Zürich
Bern
SWITZ.
Geneva
Vaduz
Bay of Biscay
FRANCE
Lyon
Ljubljana
SLOVENIA
Zagreb
Bucharest
Constanța
A Coruña
Bordeaux
Milan
Venice
Turin
Genoa
CROATIA
BOSNIA AND HERZEGOVINA
Sarajevo
Belgrade
SERBIA AND MONTENEGRO
Varna
BULGARIA
Sofia
Bilbao
Toulouse
MONACO
SAN MARINO
Florence
Andorra la Vella
ANDORRA
Marseille
Ligurian Sea
ITALY
Adriatic Sea
Skopje
F.Y.R.O.M.
Istanbul
Porto
Zaragoza
Corsica
Rome
Tirana
ALB.
Thessaloniki
PORTUGAL
Madrid
Barcelona
VATICAN CITY
Lisbon
SPAIN
Valencia
Balearic Sea
Sardinia
Naples
Tyrrhenian Sea
GREECE
Aegean Sea
Athens
BALEARIC ISLANDS
Sevilla
Cagliari
Ionian Sea
Mediterranean Sea
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

1. **Motivation**
   - Why variance and the quantum Fisher information is important?
2. **Variance and quantum Fisher information**
   - Basic definitions
   - Entanglement detection with the variance
   - Entanglement detection with the quantum Fisher information
3. **Generalized variance and quantum Fisher information**

Why variance and the quantum Fisher information is important?

- Variance is a quantity appearing often in all areas of physics.
- Quantum Fisher information is an important notion in metrology. Any connection between the two is interesting.
- Concave roofs, convex roofs are also interesting - the are typically difficult to compute.

Outline

Variance

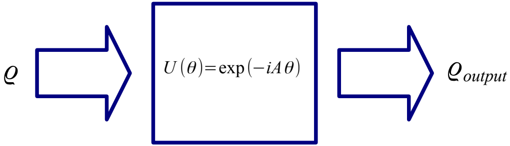
- The variance is defined as

$$(\Delta A)^2{}_{\varrho} = \langle A^2 \rangle_{\varrho} - \langle A \rangle_{\varrho}^2.$$

- The variance is concave.

Quantum Fisher information

- The small parameter θ must be estimated by making measurements on the output state :



- Cramér-Rao bound

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q^{\mathrm{usual}}[\varrho, A]}}.$$

- The quantum Fisher information is

$$F_Q^{\mathrm{usual}}[\varrho, A] = 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |A_{ij}|^2.$$

- For pure states, $F_Q^{\mathrm{usual}}[\varrho, A] = 4(\Delta A)^2_{\varrho}$, and it is convex.

Outline

Entanglement detection with the variance

- For a product states $\varrho = \varrho_1 \otimes \varrho_2$ we have

$$(\Delta(A_1 \otimes \mathbb{1} + \mathbb{1} \otimes A_2))^2{}_{\varrho} = (\Delta A_1)^2{}_{\varrho_1} + (\Delta A_2)^2{}_{\varrho_2}.$$

  Here, A_1 and A_2 act on the first and second subsystem, respectively.

- B_1 and B_2, acting on the same subsystems. They fulfill the uncertainty relations

$$(\Delta A_k)^2{}_{\varrho_k} + (\Delta B_k)^2{}_{\varrho_k} \geq L_k,$$

  where L_k are some constants.

- Hence, for product states

$$(\Delta(A_1 \otimes \mathbb{1} + \mathbb{1} \otimes A_2))^2{}_{\varrho} + (\Delta(B_1 \otimes \mathbb{1} + \mathbb{1} \otimes B_2))^2{}_{\varrho} \geq L_1 + L_2.$$

- Due to convexity, also true for separable states.

[O. Gühne, Phys. Rev. Lett. 92, 117903 (2004).]

Entanglement detection with the variance II

- Note that only two properties of the variance were used:
  - For pure states, it is $\langle A^2 \rangle_\Psi - \langle A \rangle_\Psi^2$.
  - It is concave.
- Another quantity with these properties could also be used for entanglement detection.
- If it were smaller than the variance, then it would even be better than the variance for this purpose.

Outline

Entanglement detection with the q. Fisher information

- For our bipartite system, for pure states we have

$$(\Delta A_k)^2{}_{\varrho_k} \leq M_k,$$

where M_k are some constants.

- Based on these, for pure product state we have

$$(\Delta(A_1 \otimes \mathbb{1} + \mathbb{1} \otimes A_2))^2{}_{\varrho} \leq M_1 + M_2,$$

- Then, due to the convexity of the quantum Fisher information, for mixed separable states we have.

$$F_Q^{\mathrm{usual}}[\varrho, A_1 \otimes \mathbb{1} + \mathbb{1} \otimes A_2] \leq 4(M_1 + M_2).$$

Any state that violates this is entangled.

[L. Pezze and A. Smerzi , Phys. Rev. Lett. 102, 100401 (2009).]

Entanglement detection with the q. Fisher information II

- Note that only two properties of the quantum Fisher information were used:
  - For pure states, it is $\langle A^2\rangle_\Psi - \langle A\rangle_\Psi^2$.
  - It is convex.
- Another quantity with these properties could also be used for entanglement detection.
- If it were larger than the usual quantum Fisher information, then it would even be better for this purpose.

Generalized variance

Definition 1. Generalized variance $\mathrm{var}_\varrho(A)$ is defined as follows.

1. For pure states, we have
$$\mathrm{var}_\Psi(A) = (\Delta A)^2{}_\Psi.$$
2. For mixed states, $\mathrm{var}_\varrho(A)$ is concave in the state.

Definition 2. The minimal generalized variance $\mathrm{var}_\varrho^{\min}(A)$ is defined as follows.

1. For pure states, it equals the usual variance
$$\mathrm{var}_\Psi^{\min}(A) = (\Delta A)^2{}_\Psi,$$
2. For mixed states, it is defined through a concave roof construction
$$\mathrm{var}_\varrho^{\min}(A) = \sup_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k},$$
where
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

Theorem 1

Theorem 1. The minimal generalized variance is the usual variance

$$\mathrm{var}_{\varrho}^{\min}(A) = (\Delta A)^2{}_{\varrho}.$$

In other words, the variance its own concave roof.

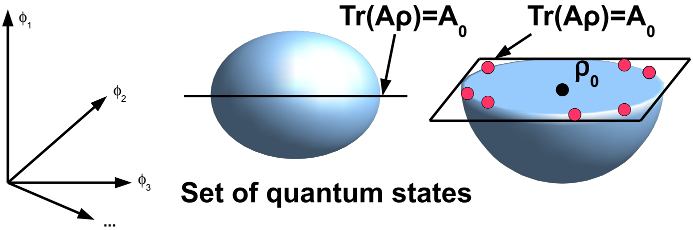
Handwaving proof:

$$(\Delta A)^2_{\varrho} = \sum_k p_k (\Delta A)^2{}_{\Psi_k} + (\langle A\rangle_{\Psi_k} - \langle A\rangle_{\varrho})^2.$$

You can always find a decomposition such that $\langle A\rangle_{\Psi_k} = \langle A\rangle_{\varrho}$ for all k.

Theorem 1

Handwaving proof, continuation, Geometric argument:



For details, please see arxiv:1109.2831.

Generalized quantum Fisher information

Definition 3. The generalized quantum Fisher information $F_Q[\varrho, A]$ is defined as follows.

1. For pure states, we have

$$F_Q[\varrho, A] = 4(\Delta A)^2{}_\Psi.$$

   The factor 4 appears for historical reasons.

2. For mixed states, $F_Q[\varrho, A]$ is convex in the state.

Definition 4. $F_Q^{\max}[\varrho, A]$ is defined as follows.

1. For pure states, it equals four times the usual variance

$$F_Q^{\max}[\varrho, A] = 4(\Delta A)^2{}_\Psi.$$

2. For mixed states, it is defined through a convex roof construction

$$F_Q^{\max}[\varrho, A] = 4 \inf_{\{p_k, \Psi_k\}} p_k (\Delta A)^2{}_{\Psi_k}.$$

Theorem 2

Theorem 2. The maximal generalized quantum Fisher information is the usual quantum Fisher information for rank-2 states.

$$F_Q^{\text{max}}[\varrho, A] = F_Q^{\text{usual}}[\varrho, A]$$

In other words, the quantum Fisher information is the convex roof of the variance for rank-2 states.

It would be interesting to find connection to the statements of
[B.M. Escher,R.L.de Matos Filho, and L. Davidovich, Nature Phys. (2011)]
concerning quantum Fisher information and purifications.
(For the idea, thanks to Rafal Demkowicz-Dobrzanski.)

Generalized variance and quantum Fisher information in the literature

- D. Petz defined before generalized variances and quantum Fisher informations.

- He presents formulas, that define a variance and a corresponding quantum Fisher information for each each standard matrix monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$.

- Surprisingly, his variances and quantum Fisher informations definitions fit the definitions of this presentation.

- D. Petz, *Quantum Information Theory and Quantum Statistics* (Springer-Verlag, Heidelberg, 2008).
- D. Petz, J. Phys. A: Math. Gen. **35**, 79 (2003).
- P. Gibilisco, F. Hiai and D. Petz, IEEE Trans. Inform. Theory **55**, 439 (2009).
- F. Hiai and D. Petz, From quasi-entropy, http://arxiv.org/abs/1009.2679.

Summary

- We discussed how to define the generalized variance and the generalized quantum Fisher information.
- We found that the variance its own concave roof, while the quantum Fisher information is its own convex roof for rank-2 states.

See:
G. Tóth and D. Petz,
Optimal generalized variance and quantum Fisher information,
arxiv:1109.2831.

THANK YOU FOR YOUR ATTENTION!

European Research Council

MINISTERIO DE CIENCIA E INNOVACIÓN

EUSKO JAURLARITZA GOBIERNO VASCO