Genuine three-partite entangled states with a hidden variable model

Géza Tóth[1,2] and Antonio Acín[3]

[1]*Max-Planck Institute for Quantum Optics, Garching, Germany*

[2]*Research Institute for Solid State Physics and Optics, Budapest, Hungary*

[3]*ICFO, Barcelona, Spain*

quant-ph/0512088

Motivation

Bell inequalities have already been used many times for detecting entangled quantum states which *do not allow* for a local hidden variable (LHV) model.

In our work, we would like to find entangled quantum states which *allow* for a local description, in the multipartite scenario. Even in the bipartite case, there is very little literature on that. That is, we know that LHV models exits for some entangled states, but we know very little about how they look like.

Outline

- Local hidden variable models
- Mixtures and quasi mixtures
- Alternative derivation for two-qubit Werner states
- Three-qubit state with a hidden variable model

- Local hidden variable models
- Mixtures and quasi mixtures
- Alternative derivation for two-qubit Werner states
- Three-qubit state with a hidden variable model

Local hidden variable models

Consider bipartite systems, with parties A and B. For a quantum state there is a LHV model describing any von Neumann measurement if for any M_A/M_B there are

$$\langle M_A \rangle_\omega \quad \text{and} \quad \langle M_B \rangle_\omega$$

such that two-body correlations are given as

$$\langle M_A \otimes M_B \rangle = \int M(d\omega) \langle M_A \rangle_\omega \langle M_B \rangle_\omega .$$

Here ω is the hidden variable.

Local hidden variable models II

For all separable state there it is trivial to find such a hidden variable model.

Werner presented a LHV model in 1989 for two-qubit states of the form

$$\rho_W = p\,|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{1}}{4}; \quad |\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}\ .$$

These states are Werner states. Similar definition exists also for higher dimensional bipartite systems (qutrits, qudits).

Interesting property of these states is that they are invariant under transformations of the type $U \otimes U$.

- Local hidden variable models
- Mixtures and quasi mixtures
- Alternative derivation for two-qubit Werner states
- Three-qubit state with a hidden variable model

Quasi mixtures I.

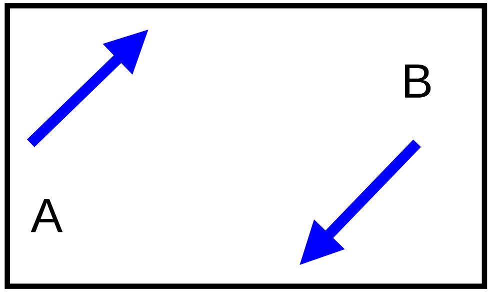
Let us consider the following mixture

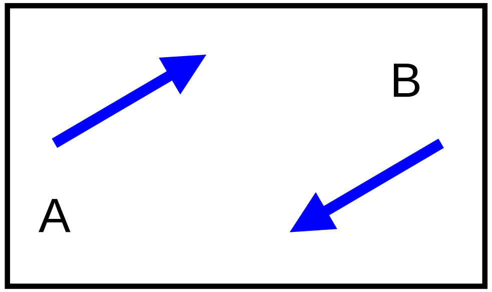
$$\rho := \int_{\omega \in C(2), |\omega|=1} M(d\omega)\tau_\omega \otimes |\omega\rangle\langle\omega|,$$

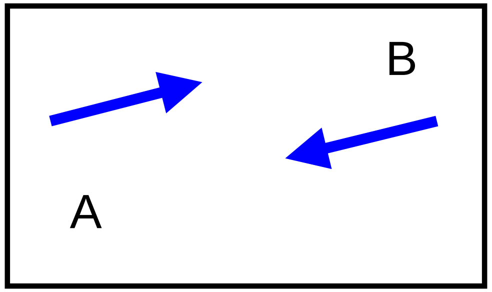
where

$$\tau_\omega := \frac{1}{2}\left(1 - \sum_{k=x,y,z} \langle\sigma_k\rangle_\omega \sigma_k\right).$$

State ρ is obviously $U \otimes U$ invariant and direct computation shows that it is positive semi-definite. It is actually a *non-entangled* Werner state.







Quasi mixtures II.

How can we modify by choosing a different τ_ω this mixture such that we get entangled Werner states? Clearly, τ_ω must be aphysical. One such choice is

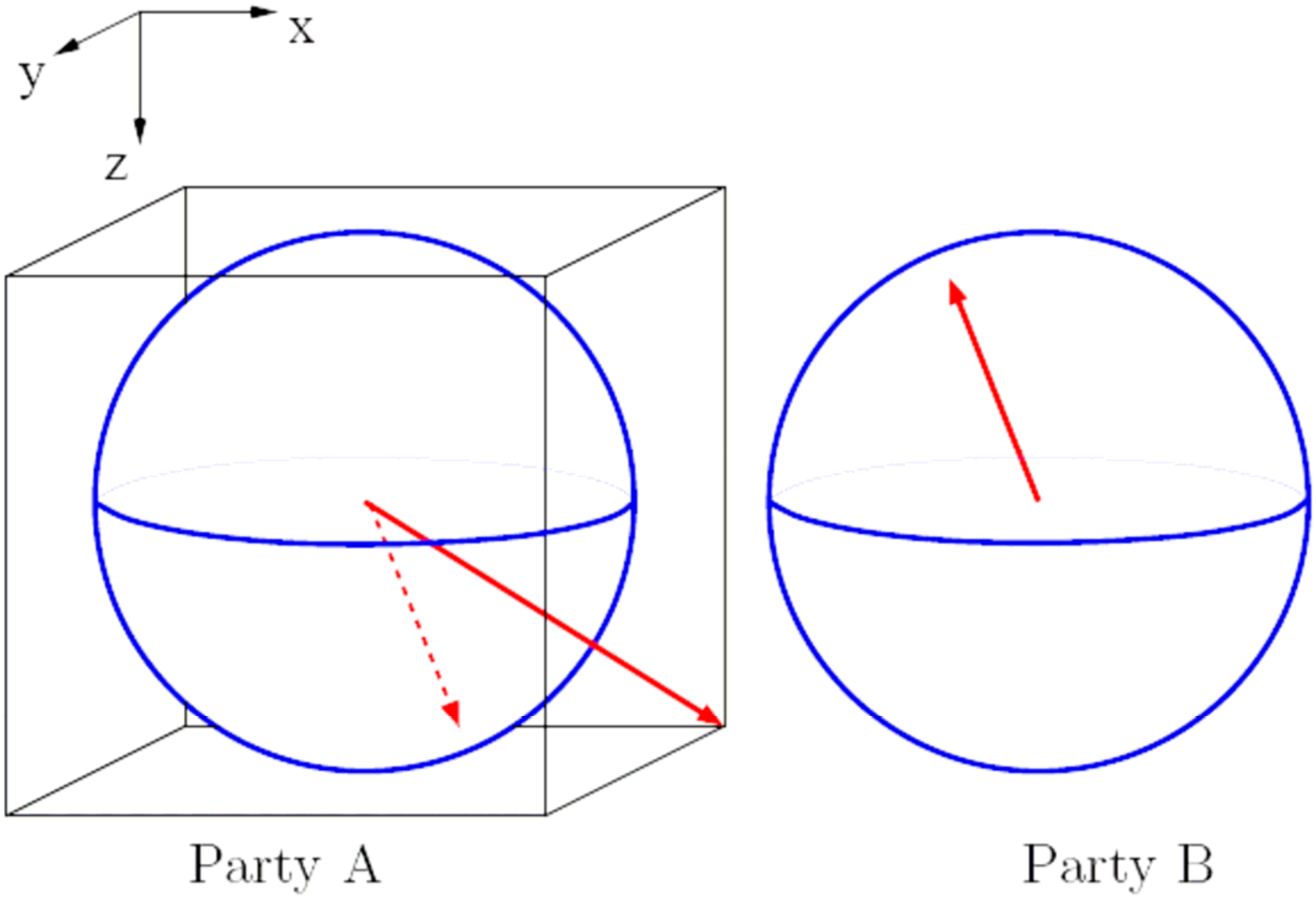
$$\hat{\tau}_\omega := \frac{1}{2}\left(1 - \sum_{k=x,y,z} sign(\langle \sigma_k \rangle_\omega)\sigma_k\right).$$

Direct calculation shows that

$$\rho := \int_{\omega \in C(2)} M(d\omega)\hat{\tau}_\omega \otimes |\omega\rangle\langle\omega|$$

is an entangled Werner state with *p*=1/2.

Quasi mixtures III



- Local hidden variable models
- Mixtures and quasi mixtures
- Alternative derivation for two-qubit Werner states
- Three-qubit state with a hidden variable model

Alternative derivation

Here we present an alternative derivation of the LHV model for two-qubit Werner states. Let us look again the decomposition of ρ as a quasi mixture

$$\rho := \int_{\substack{\omega\in C(2),\\ |\omega|=1}} M(d\omega)\frac{1}{2}\left(1-\sum_{k=x,y,z} sign\left(\langle\sigma_k\rangle_\omega\right)\sigma_k\right)\otimes|\omega\rangle\langle\omega|.$$

This immediately provides a LHV model for measuring $M_A = \pm\sigma_{x/y/z}$ on party A and measuring arbitrary M_B for party B by choosing

$$\langle M_A\rangle_\omega = Tr(\frac{1}{2}\left(1-\sum_{k=x,y,z} sign\left(\langle\sigma_k\rangle_\omega\right)\sigma_k\right)M_A),$$

$$\langle M_B\rangle_\omega = Tr(|\omega\rangle\langle\omega|M_B).$$

Alternative derivation II

What if M_A is not $\sigma_{x/y/z}$? Then $\left\langle M_A \right\rangle_\omega$ would be larger than 1 for some omega. Thus, we do not have a valid LHV model for arbitrary measurement on M_A.

Let us see how to make an LHV model for an arbitrary M_A with a +1/-1 spectrum. At this point we can exploit that our state has a $U \otimes U$ symmetry. Due to this

$$\langle M_A \otimes M_B \rangle \ = \ \langle \sigma_z^{(A)} \otimes M_B' \rangle.$$

Let us define U_A as

$$M_A = U_A^\dagger \sigma_z U_A.$$

Then

$$M_B' \ = \ U_A M_B U_A^\dagger.$$

If we do not have a LHV model for M_A and M_B, it is enough to have one For σ_z and M_B'.

Alternative derivation III

Thus we *almost* have a model for arbitrary M_A and M_B

$$\langle M_A \rangle_\omega = \mathrm{Tr}(\sigma_z \varrho_\omega) = -\mathrm{sign}[\mathrm{Tr}(\sigma_z |\omega\rangle\langle\omega|)],$$
$$\langle M_B \rangle_\omega = \mathrm{Tr}(M'_B \rho_\omega) = \mathrm{Tr}(U_A M_B U_A^\dagger |\omega\rangle\langle\omega|).$$

However, $\langle M_B \rangle_\omega$ depends on U_A. This can be solved by introducing

$$|\omega'\rangle = U_A^\dagger |\omega\rangle \, .$$

Finally, the LHV model is

$$\boxed{\begin{aligned} \langle M_A \rangle_{\omega'} &= -\mathrm{sign}[\mathrm{Tr}(M_A |\omega'\rangle\langle\omega'|)] \, , \\ \langle M_B \rangle_{\omega'} &= \mathrm{Tr}(M_B |\omega'\rangle\langle\omega'|). \end{aligned}}$$

Three-qubit case

Let us now consider the three-qubit state

$$\rho_3 := \int_{\substack{\omega\in C(2),\\ |\omega|=1}} M(d\omega)\frac{1}{2}\left(1-\sum_{k=x,y,z} sign\left(\langle\sigma_k\rangle_\omega\right)\sigma_k\right)\otimes|\omega\rangle\langle\omega|\otimes|\omega\rangle\langle\omega|.$$

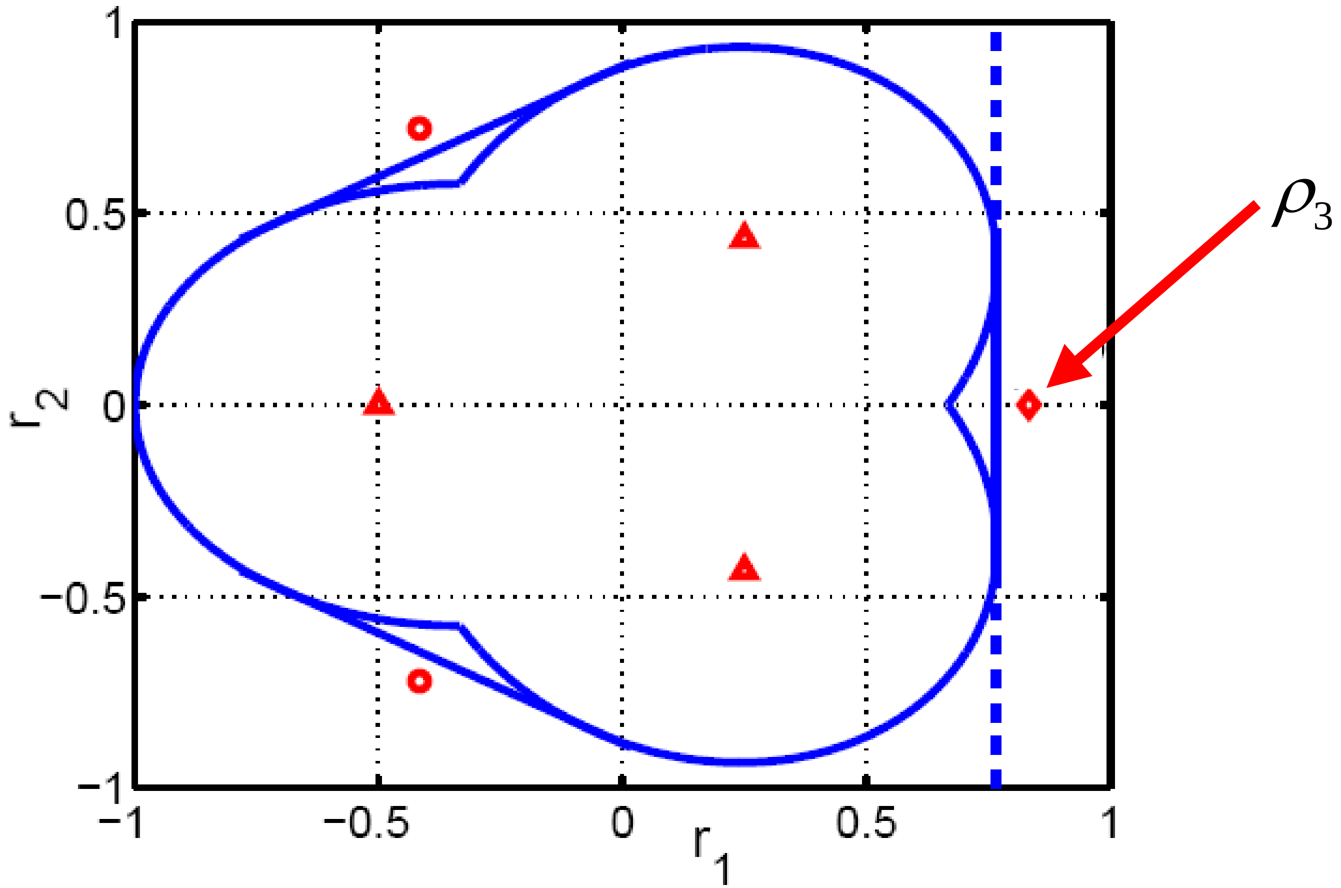
Direct computation shows that it is a physical density matrix (positive semi-definite) and it is $U\otimes U\otimes U$ invariant. Based on the previous ideas it also follows that all three-body correlations can be obtained from a local model.

The model is

$$\begin{aligned}
\langle M_A\rangle_\omega &= -\mathrm{sign}[\mathrm{Tr}(M_A|\omega\rangle\langle\omega|)],\\
\langle M_B\rangle_\omega &= \mathrm{Tr}(M_B|\omega\rangle\langle\omega|),\\
\langle M_C\rangle_\omega &= \mathrm{Tr}(M_C|\omega\rangle\langle\omega|).
\end{aligned}$$

Three-qubit entanglement

Our state is genuine three-qubit entangled



Here r_1 and r_2 are expectation values of some operators (see paper). The three disks with their convex hull are the states without three-qubit entanglement. (See also Eggeling & Werner, PRA 2001).

Our state is fully distillable

$$\rho_3 := \int_{\substack{\omega\in C(2),\\ |\omega|=1}} M(d\omega)\frac{1}{2}\left(1-\sum_{k=x,y,z} sign\left(\langle\sigma_k\rangle_\omega\right)\sigma_k\right)\otimes|\omega\rangle\langle\omega|\otimes|\omega\rangle\langle\omega|.$$

The reduced density matrices for AB and for AC are

$$\rho_{AC} = \rho_{AB} = \int_{\substack{\omega\in C(2),\\ |\omega|=1}} M(d\omega)\frac{1}{2}\left(1-\sum_{k=x,y,z} sign\left(\langle\sigma_k\rangle_\omega\right)\sigma_k\right)\otimes|\omega\rangle\langle\omega|.$$

That is

$$\rho_{AB} = \rho_{AC} = \rho_W$$

and the reduced matrices are entangled. Thus a singlet can be distilled between A and B, and between A and C. Hence any three-qubit state can be obtained from many copies of our state with local operations and classical communications.

Conclusions

We have presented a three-qubit quantum state. Any von Neumann measurement on this state can be described by a local hidden variable model. This state is, surprisingly, genuine three-qubit entangled.

For more details, for example, LHV models for higher dimensions and higher number of qudits see

quant-ph/0512088.

For further information please see

http://optics.szfki.kfki.hu/~toth