Two-setting Bell inequalities for graph states

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G. Tóth, O. Gühne, and H.J. Briegel, PRA **73**, 022303 (2006).

Motivation and Abstract

Cluster states and graph states appear very often in quantum information, e.g., in error correction, measurement-based quantum computing, etc.

It is important to know, how much noise we can mix with a graph state such that the state is still violates a Bell inequality, i.e., it is still non-local. (States violating a Bell inequality are more useful than the ones which do not.)

We will find Bell inequalities which are maximally violated by graph states, and the violating increases exponentially with the size for some type of graph states. Our inequalities need the measurement of at most two operators per site.

Outline

- Stabilizer theory
- Previous work
- Basic inequality
- Composite inequality with a violation increasing with the size

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Stabilizer theory

Definition: A quantum state $|\Psi\rangle$ is stabilized by an operator *S* if

$$S|\Psi\rangle = |\Psi\rangle.$$

In other words, *S* is the stabilizing operator of $|\Psi\rangle$.

The key idea is that an *N*-qubit quantum state can uniquely be defined by *N* stabilizing operators. For certain quantum states these operators are very simple ...

Stabilizer theory is used in quantum error correction and fault tolerant quantum computation.

D. Gottesmann, PRA **54**, 1862 (1996).

GHZ states

Generalized *N*-qubit GHZ state:

$$|GHZ_N\rangle = \frac{1}{\sqrt{2}}\left(|000...00\rangle + |111...11\rangle\right)$$

Stabilizing operators of a GHZ state:

$$g_1^{(GHZ_N)} = X^{(1)}X^{(2)}\cdots X^{(N)},$$

$$g_2^{(GHZ_N)} = Z^{(1)}Z^{(2)},$$

$$g_3^{(GHZ_N)} = Z^{(2)}Z^{(3)}, \qquad \ldots$$

$$g_N^{(GHZ_N)} = Z^{(N-1)}Z^{(N)}.$$

Not only these operators, but also their products stabilize the GHZ state. These form a group called *stabilizer*. g_k's are the generators of the stabilizer.

GHZ states – The stabilizer group

Three-qubit example: the stabilizer group has 8 elements

Generators:

$$X^{(1)}X^{(2)}X^{(3)}$$

$$Z^{(1)}Z^{(2)}$$

$$Z^{(2)}Z^{(3)}$$

Obtained from products of the generators:

$$-Y^{(1)}X^{(2)}Y^{(3)}$$

$$-Y^{(1)}Y^{(2)}X^{(3)}$$

$$-X^{(1)}Y^{(2)}Y^{(3)}$$

$$Z^{(1)}Z^{(3)}$$

$$1$$

Cluster states

Cluster states appear in

- error correction, fault tolerant quantum computing, and
- Naturally arise in spin chains with nn Ising coupling.

Generators of the stabilizer for the *N*-qubit cluster state $|LC_N\rangle$

$$g_k^{(LC_N)} = Z^{(k-1)} X^{(k)} Z^{(k+1)},$$

where *k*=1,2,...,*N* and

$$Z^{(0)} = Z^{(N+1)} = 1.$$

R. Raussendorf and H.J. Briegel, PRL **86**, 5188 (2001).
See also recent experiments with a four-qubit cluster state with photons in Vienna (Nature) and at MPQ, Garching (PRL).

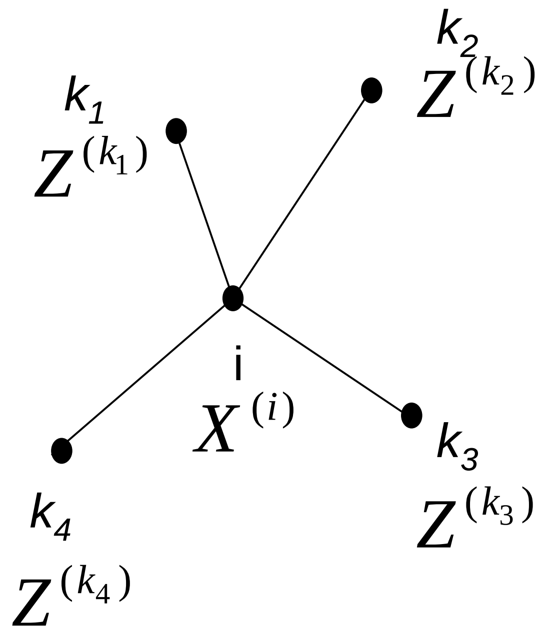
Graph states

Graph states are generalizations of cluster states. They are defined by a graph of *N* vertices. Some of these vertices are connected by each other by an edge.

Physical meaning of vertices: The graph state can be created with an Ising interaction between the connected qubits.

Let us denote by $\mathcal{N}(i)$ the set of vertices connected to vertex *i*. Then the generators of the stabilizer for a graph state are

$$g_i^{(G_N)} = X^{(i)} \prod_{k \in \mathcal{N}(i)} Z^{(k)}.$$



Note that there is a generator for each qubit.

These define the graph state uniquely.

* Cluster states are graph states with a linear graph.
* GHZ states are graph states with a „star“ graph.

- Stabilizer theory (GHZ and cluster states)
- Previos work
- Finding a lower bound on the fidelity with few measurements
- Detecting genuine multi-qubit entanglement with few measurements
- Related issues: Bell inequalities, energy as a witness, etc.

Mermin inequality

Bell inequalities for GHZ states (Mermin 1990). The Bell operator is

$$B := X_1 X_2 X_3 X_4 \ldots - Y_1 Y_2 X_3 X_4 \ldots + Y_1 Y_2 Y_3 Y_4 \ldots - + \ldots \leq 2^{N/2}$$

Here each term represents the sum of its all possible permutations. This Bell inequality is maximally violated by GHZ states.

Written with stabilizing operators:

$$B := g_1^{(GHZ)} \prod_{k=2}^{n} (1 + g_k^{(GHZ)})$$

Key observation!!!

Stabilizer states

Contradictions obtained from assuming local realism for a multi-qubit quantum code word.

D. P. DiVincenzo and A. Peres, Quantum code words contradict local realism**,** PRA **55**, 4089 (1997).

Four-qubit cluster state

Non-locality of graph states and a Bell inequality for 4-qubit graph states

$$X_1 X_3 Z_4 + Z_1 Y_2 Y_3 Z_4 + X_1 Y_3 Y_4 - Z_1 Y_2 X_3 Y_4 \leq 2$$

The maximum for quantum states is 4. This inequality needs the measurement of two observables on qubits 1,3 and 4, while for qubit 2 only a single observable is measured.

It is a three-body Mermin inequality with multi-qubit variables.

V. Scarani, A. Acín, E. Schenck, and M. Aspelmeyer, PRA **71**, 042325 (2005).

Inequalities with stab. operators

The inequalities for any graph state are constructed from all the elements of the stabilizer.

Idea: Our Bell operator for detecting a state Ψ looks like

$$B = 2^N \left|\Psi\right\rangle\left\langle\Psi\right| = \sum_{k=1}^{2^N} S_k$$

With this a Bell inequality can be obtained as

$$\left\langle B \right\rangle \leq const.$$

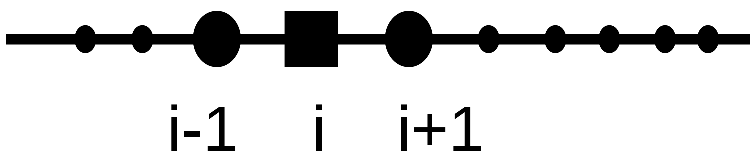
O. Gühne, G. Tóth, P. Hyllus, and H.J. Briegel, PRL **95**, 120405 (2005).

- Stabilizer theory
- Previous work
- Basic inequality
- Composite inequality with violation increasing with the size

Basic inequality

Our aim is to construct a Bell inequality which does not use all the elements of the stabilizer.

Let us consider now an *N*-qubit cluster state and write down an inequality which is maximally violated by it. We use our „key observation“ for the case of GHZ states.



$$
\begin{aligned}
\mathcal{B}_i^{(LC_N)} &:= g_i^{(LC_N)}(1+g_{i+1}^{(LC_N)})(1+g_{i-1}^{(LC_N)}) \\
&= Z^{(i-1)}X^{(i)}Z^{(i+1)} + Z^{(i-2)}Y^{(i-1)}Y^{(i)}Z^{(i+1)} \\
&+ Z^{(i-1)}Y^{(i)}Y^{(i+1)}Z^{(i+2)} \\
&- Z^{(i-2)}Y^{(i-1)}X^{(i)}Y^{(i+1)}Z^{(i+2)},
\end{aligned}
$$

Maximum for quantum states is 4, maximum for local hidden variable models is 2.

Basic inequality – General case

Let us consider now consider an N-qubit graph state. The generators of the stabilizer are g_1, g_2, ..., g_n. We want to construct a Bell inequality which is maximally violated by this graph state $|G_N\rangle$.

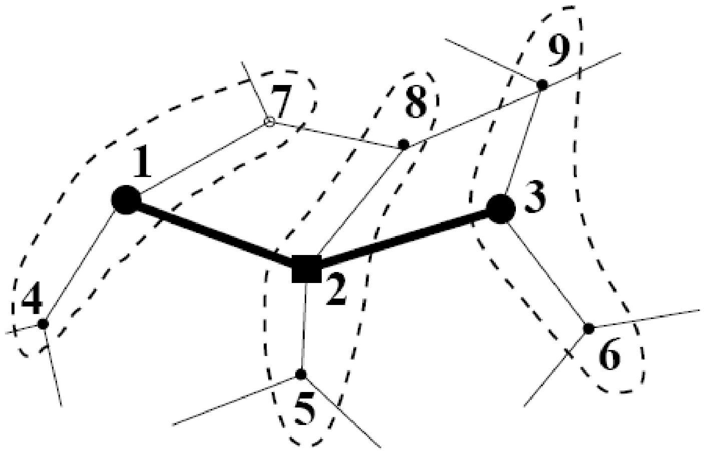
Theorem 1. Let i be a vertex and let $I \subseteq \mathcal{N}(i)$ be a subset of its neighborhood, such that none of the vertices in I are connected by an edge. Then the following operator

$$\mathcal{B}(i, I) := g_i \prod_{j \in I} (1 + g_j), \qquad (10)$$

can be used as a Bell operator of a Bell inequality. For the maximum for local models see G. Tóth, O. Gühne, and H.J. Briegel, PRA **73**, 022303 (2006).

Basic inequality – General case II

Example: Let us consider the graph state given by the graph



Then we can write down a Bell operator involving g_1, g_2 and g_3

$$\begin{aligned}\mathcal{B}(2,\{1,3\}) = {} & Z^{(1)}(Z^{(5)}Z^{(8)}X^{(2)})Z^{(3)} \\ & +(Z^{(4)}Z^{(7)}Y^{(1)})(Z^{(5)}Z^{(8)}Y^{(2)})Z^{(3)} \\ & +Z^{(1)}(Z^{(5)}Z^{(8)}Y^{(2)})(Z^{(6)}Z^{(9)}Y^{(3)}) \\ & -(Z^{(4)}Z^{(7)}Y^{(1)})(Z^{(5)}Z^{(8)}X^{(2)})(Z^{(6)}Z^{(9)}Y^{(3)}).\end{aligned}$$

Bound for local models

The maximum for the Bell operator for local models can be found since our Bell operator is the same as the Bell operator of the Mermin inequality *with multi-qubit observables*. Since we know the maximum for the Mermin inequality, we also know the maximum for our inequality.

- Stabilizer theory
- Previous work
- Basic inequality
- Composite inequality with violation increasing with the size

Composite inequality

Let us assume that we have two Bell inequalities

$$|\mathcal{E}_1| \leq \mathcal{C}_1,$$
$$|\mathcal{E}_2| \leq \mathcal{C}_2.$$

Then multiplying them new Bell inequalities are obtained

$$|\mathcal{E}_1 \mathcal{E}_2| \leq \mathcal{C}_1 \mathcal{C}_2.$$

However, one must be careful: In the composite inequalities there should not be terms like $X_1 Y_1$. That is, in a correlation term there can be only a single variable for a qubit.

Composite inequalities

Let us consider a Bell operator for the linear cluster state involving the stab. operator of qubits *i*-1, *i* and *i*+1:

$$
\begin{aligned}
\mathcal{B}_i^{(LC_N)} &:= g_i^{(LC_N)}(1+g_{i+1}^{(LC_N)})(1+g_{i-1}^{(LC_N)}) \\
&= Z^{(i-1)}X^{(i)}Z^{(i+1)} + Z^{(i-2)}Y^{(i-1)}Y^{(i)}Z^{(i+1)} \\
&+ Z^{(i-1)}Y^{(i)}Y^{(i+1)}Z^{(i+2)} \\
&- Z^{(i-2)}Y^{(i-1)}X^{(i)}Y^{(i+1)}Z^{(i+2)}, \\
\mathcal{V}(\mathcal{B}_i^{(LC_N)}) &= 2. \qquad (19)
\end{aligned}
$$

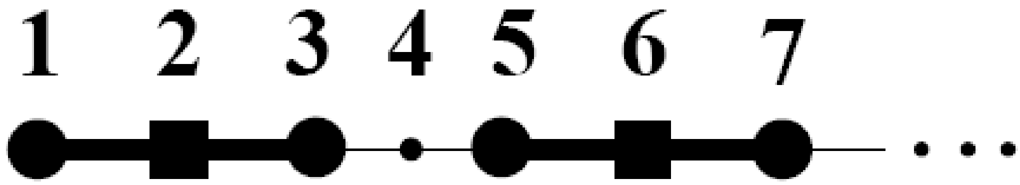
Then a composite inequality can be constructed

$$
\begin{aligned}
\mathcal{B}^{(LC_N)} &:= \prod_{i=1}^{N/4} \mathcal{B}_{4i-2}^{(LC_N)}, \\
\mathcal{V}(\mathcal{B}^{(LC_N)}) &= 2^{N/4}.
\end{aligned}
$$

The relative violation increases exponentially with *N*.

Composite inequalities II

Graphic representation of the composite inequality:



$B(2,\{1,3\})$ $B(6,\{5,7\})$

GHZ-type violation of locality

For states with a local hidden variable model (Mermin 1990) :

$$\left\langle X^{(1)} X^{(2)} X^{(3)} \right\rangle - \left\langle Y^{(1)} Y^{(2)} X^{(3)} \right\rangle - \left\langle X^{(1)} Y^{(2)} Y^{(3)} \right\rangle - \left\langle Y^{(1)} X^{(2)} Y^{(3)} \right\rangle \leq 2.$$

For the GHZ state all terms are +1 and the left hand side is 4.

Our inequalities are also like that: All correlation terms are +1 for a graph state. Thus our inequalities are violated in the Greenberger-Horne-Zeilinger sense.

Conclusions

We have discussed how to create a Bell inequality for a given graph state which is maximally violated only by this state. First a basic inequality was presented involving only some of the qubits. Then we saw how to create composite inequalities for which the degree of violation increases exponentially with the size.

For further information please see

G. Tóth, O. Gühne, and H.J. Briegel, PRA **73**, 022303 (2006)

and

http://optics.szfki.kfki.hu/~toth.