Multi-party entanglement detection in spin chains and optical lattices of two-state bosonic atoms with collective measurements

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Outline

Motivation – The definition of collective measurement.

Criterion I: Detects entangled states close to a cluster
state with an entanglement witness.

Criterion II: Similar as the previous, but it uses an uncertainty
relation instead of an entanglement witness.

Criterion III: Detects states close to a many-body singlet
using uncertainty relations

Objectives

We would like to detect entanglement in a spin chain or in a 1D lattice of two-state bosonic atoms without the possibility of individual access to spins/lattice sites.

1. We can measure only collective quantities, i.e., the sum of single system measurement results.

2. We can execute only the same single qubit operation on all the qubits.

3. We have a simple nearest-neighbor interaction, which is uniform over the lattice.

What can be measured in this system?

x/y/z components of the collective angular momentum

$$J_{x/y/z} = \sum_k \sigma_{x/y/z}^{(k)}$$

Knowing $\langle J_x \rangle$, $\langle J_y \rangle$ and $\langle J_z \rangle$ is not enough for entanglement detection since for arbitrary values of these three there is a corresponding separable state.

We have two choices:

(i) We need a multi-qubit dynamics before measuring $\langle J_{x/y/z} \rangle$

(ii) We need higher moments of $J_{x/y/z}$

Criterion 1: detecting the cluster state with a preceding quantum dynamics I

Method: entanglement witness #atoms/site:1

Necessary condition for separability for a spin chain with an entanglement witness:

$$J = \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle \leq \frac{N}{2}$$

Later it will be explained how to measure the third order correlation terms with dynamics + a simple collective measurement.

The criterion is maximally violated for cluster states.

Proof of Criterion I

Consider the following sum of two consecutive terms:

$$J_k = \left\langle \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} + \sigma_z^{(k)} \sigma_x^{(k+1)} \sigma_z^{(k+2)} \right\rangle \quad (*)$$

For a product state:

$$J_k = \left\langle \sigma_z^{(k-1)} \right\rangle \left\langle \sigma_x^{(k)} \right\rangle \left\langle \sigma_z^{(k+1)} \right\rangle + \left\langle \sigma_z^{(k)} \right\rangle \left\langle \sigma_x^{(k+1)} \right\rangle \left\langle \sigma_z^{(k+2)} \right\rangle$$
$$\leq \left\langle \sigma_x^{(k)} \right\rangle \left\langle \sigma_z^{(k+1)} \right\rangle + \left\langle \sigma_z^{(k)} \right\rangle \left\langle \sigma_x^{(k+1)} \right\rangle \leq 1$$

If it is true for a product state, then it is also true for an separable state since (*) is linear in operator expectation values.

For an entangled state: $J_k \leq 2$

End of Proof for Criterion I

For a separable quadruplet

$$J_k \leq 1$$

J_k involves spin *(k-1), (k), (k+1)* and *(k+2).* Hence

$$J = \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle = \sum_{k=1,3,5,\ldots} J_k \leq \frac{N}{2}$$

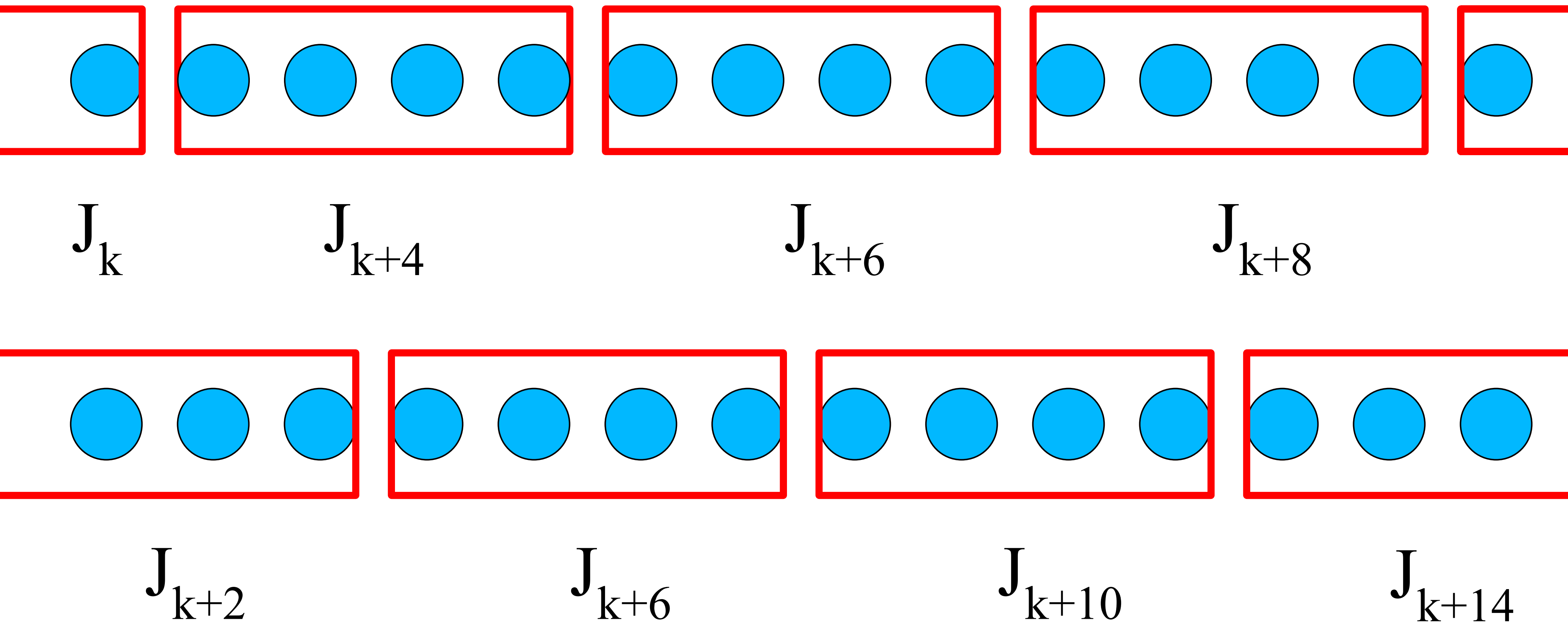
Q.E.D. Comment: for an entangled state

$$J \leq N$$

The number of entangled quadruplets is proportional to *(J-N/2).*

Entanglement quantification: overlapping quadruplets

$$J=\left\langle \sum_k \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)} \right\rangle = \sum_{k=1,3,5,\ldots} J_k \le \frac{N}{2}$$



If the *kth* quadruplet is separable then $J_k<1$. If it is entangled then $J_k<2$.

Number of entangled quadruplets:

The number of entangled *overlapping* quadruplets can be deduced as

$$N_{qo} \geq J - \frac{N}{2}$$

The lower bound for the number of *non-overlapping* entangled quadruplets is half of this

$$N_q \geq \frac{J}{2} - \frac{N}{4}$$

Q.: What state do we detect?
A.: The cluster state.

The cluster state is defined through the following eigenvalue equations:

$$\sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)}\Psi_{cluster} = \lambda_k \Psi_{cluster}$$

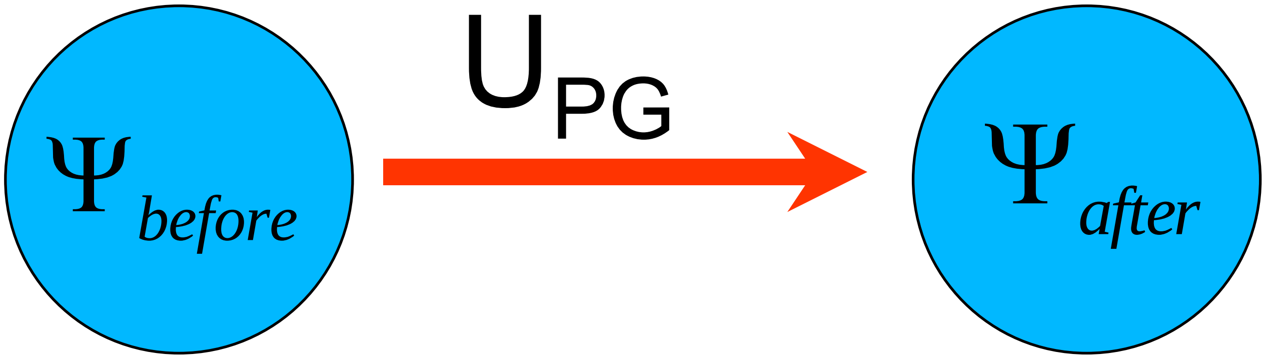
where

$$\lambda_k = \pm 1$$

With our criterion we detect cluster states with $\lambda_k = +1$.

A cluster state can be created with simple Ising chain dynamics.

How to measure *J* with preceding interaction



We would like to know whether this state is entangled

We measure J_x after executing U_{PG}

How to measure *J* with preceding interaction II

We can measure J by measuring J_x after the dynamics U_{PG}.

$$J = \left\langle \sum_k \sigma_x^{(k)} \right\rangle_{after} = \left\langle U_{PG} \sum_k \sigma_x^{(k)} U_{PG} \right\rangle_{before} = \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle_{before}$$

U_{PG} describes the application of the phase gate for all neighboring spins.

$$U_{PG} = e^{-i\frac{\pi}{4}\sum_k \left(\sigma_z^{(k)} - 1\right)\left(\sigma_z^{(k+1)} - 1\right)}$$

Experiments with the lattice of two-state bosonic atoms

The cluster state was created experimentally at LMU (Mandel, Greiner, Widera, Rom, Hänsch, Bloch, quant-ph/0308080).

The presence of the many-body entanglement was deduced from the disappearance of the interference pattern, when the atoms corresponding to one of the two species were detected after a free expansion.

Criterion 2: detecting the cluster state with a preceding quantum dynamics

Method: uncertainties

# of atoms/site: 1

For separable states:

$$J = \left\langle \sum_k \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle \leq \frac{N}{2}$$

It follows also that:

(Similarly as in O. Gühne, quant-ph/0306194)

$$\sum_k \left\langle \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right\rangle^2 \leq \frac{N}{2}$$

$$\sum_k \left[\Delta\left(\sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)} \right) \right]^2 \geq \frac{N}{2}$$

Hence for the variances:

Criterion 2: detecting the cluster state with a preceding quantum dynamics II

Collective measurement scheme. Necessary condition for separability

$$(\Delta X_1)^2 + (\Delta X_2)^2 + (\Delta X_3)^2 \geq \frac{N}{2}$$

where

$$X_{1/2/3} = \sum_k \sigma_x^{(3k+1/2/3)}$$

Comments:

- For this scheme, one needs the partitioning of the spins.
- Interesting, since the expression is not an entanglement witness, but a criterion nonlinear in expectation values. (O. Gühne, quant-ph-0306194, H. Hofmann, PRA 68, 032103(2003) + quant-ph/0212090)

Criterion 3: detecting the many-body singlet *without* a preceding quantum dynamics, based on higher order moments

Method: uncertainties

# of atoms/site: *can vary*

Necessary condition for separability:

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}$$

The condition is maximally violated for many-body singlets with zero total angular momentum.

Criterion 3: Proof

Uncertainty relation for spin (k)

$$(\Delta j_x^{(k)})^2 + (\Delta j_y^{(k)})^2 + (\Delta j_z^{(k)})^2 \geq \frac{N_k}{2}$$

For a product state single system uncertainties add up and we obtain:

$$(\Delta \sum_k j_x^{(k)})^2 + (\Delta \sum_k j_y^{(k)})^2 + (\Delta \sum_k j_z^{(k)})^2 = \sum_k j_x^{(k)} + j_y^{(k)} + j_z^{(k)} \geq \sum_k \frac{N_k}{2} = \frac{N}{2}$$

It is easy to see that this is also true for separable states [H. Hofmann, PRA 68, 032103(2003) + quant-ph/0212090; C. Simon, D. Bouwmeester, quant-ph/0302023 + PRA]. Q.E.D.

Definition of entanglement if the particle number varies on the lattice

The $j_{x/y/z}$ operators are Schwinger type angular momentum operators defined as

$$j_x^{(k)} = \frac{1}{2}\left(a_k b_k^\dagger + a_k^\dagger b_k\right) \qquad j_y^{(k)} = \frac{-i}{2}\left(a_k b_k^\dagger - a_k^\dagger b_k\right)$$

$$j_z^{(k)} = \frac{1}{2}\left(a_k a_k^\dagger - b_k^\dagger b_k\right)$$

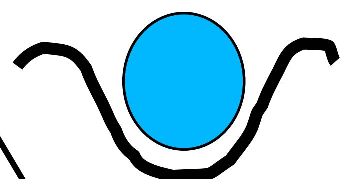
Here a_k and b_k are the destruction operators corresponding to the two internal states of the atoms.

Definition of entanglement if the particle number varies on the lattice II

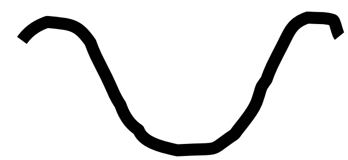
How to describe a state of a lattice site
in this *mode* picture? Let us use for labeling

$$\left| j,\ j_z \right\rangle$$

Examples:

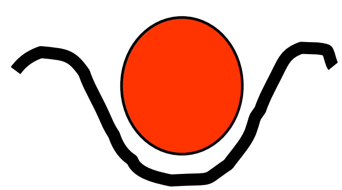


$$\left| \downarrow \right\rangle = \left| +\frac{1}{2}, -\frac{1}{2} \right\rangle$$



$$\left| 0 \right\rangle = \left| 0, 0 \right\rangle$$

$$\left| \uparrow \right\rangle = \left| +\frac{1}{2}, +\frac{1}{2} \right\rangle$$



Definition of entanglement if the particle number varies on the lattice III

Based on the previous formalism, two lattice sites can be in the state

$$\left|\downarrow\right\rangle\left|\downarrow\right\rangle+\left|\uparrow\right\rangle\left|\uparrow\right\rangle$$

which is clearly an entangled state. But the following *single particle* state looks also entangled:

$$\left|0\right\rangle\left|\uparrow\right\rangle+\left|\uparrow\right\rangle\left|0\right\rangle$$

This is a general characteristic of the “mode picture“ in contrast to the “particle picture“.

Definition of entanglement if the particle number varies on the lattice IV

Good news: our criterion does not detect the

$$\left|0\right\rangle\left|\uparrow\right\rangle+\left|\uparrow\right\rangle\left|0\right\rangle$$

state as entangled.

Reason: $J_{x/y/z}$ commute with the local particle number operators N_k.

Thus a criterion with $J_{x/y/z}$ cannot distinguish between the superposition or the mixture of states having different number of particles at the lattice sites.

Criterion 3: What kind of states do we detect?

The criterion is maximally violated for states with total angular momentum zero (many-body singlets). They are the ground states of the Hamiltonian:

$$H = J_x^{\ 2} + J_y^{\ 2} + J_z^{\ 2}$$

The ground state of the anti-ferromagnetic Heisenberg chain also gives maximal violation

$$H = \sum_k \sigma_x^{(k)} \sigma_x^{(k+1)} + \sigma_y^{(k)} \sigma_y^{(k+1)} + \sigma_z^{(k)} \sigma_z^{(k+1)}$$

Criterion 3: What kind of states do we detect? II

A spin chain can be driven into substantial violation of the inequality with a simple dynamics.

For example, a dynamics under the following Hamiltonian results in a 50% violation of the criterion for N=*6* spins

$$H = -\sum_k \sigma_x^{(k)}\sigma_x^{(k+1)} - 3\sigma_y^{(k)}\sigma_y^{(k+1)} - \sum_k \sigma_z^{(k)}$$

starting out from the initial state

$$\Psi_{initial} = \left| \uparrow\uparrow\uparrow\uparrow\uparrow \ldots \right\rangle$$

Comparison with the spin squeezing criterion

The spin squeezing criterion *(Sørensen et. al. , Nature 409, **63** (2001))* is another criterion making the detection of many-body entanglement possible with collective measurement. For separable states

$$\zeta = \frac{N\left(\Delta J_z\right)^2}{\left\langle J_x\right\rangle^2 + \left\langle J_y\right\rangle^2} \geq 1$$

Our methods have the following advantages:

● Can detect useful quantum states with J=0 (i.e., cluster state, singlet)

● Criterion III: Can distinguish between on-site and inter-site entanglement and detects only the latter

Moments of the cluster state vs. moments of the totally mixed state

The moments of J_x, J_y and J_z for a (large enough) cluster state are the same as for the totally mixed state.

Separable states can also be constructed whose second oder moments are the same as for the cluster state for any angular momentum component.

$$J_{\vec{n}} = \alpha(\vec{n}) J_x + \beta(\vec{n}) J_y + \gamma(\vec{n}) J_z$$

Thus it is hard to detect the cluster state as entangled without a preceding quantum dynamics, based only on the measurement of moments.

Summary

quant-ph/0310039

Motivation – The definition of collective measurement.

Criterion I:

- Detects entangled states close to a cluster state.
- Needs multi-qubit dynamics before measurement.
- Based on an entanglement witness.

Criterion II:

- Similar to the previous method but based on an uncertainty relati

Criterion III:

- Detects a many-body singlet using uncertainty relations.
- *Does not need* preceding quantum dynamics.

Detection of cluster state: to detect it as entangled, one needs a preceding multi-qubit dynamics.

Extra slides

Moments of the totally mixed state

The totally mixed state is defined as

$$\rho_t = \left(\left|\uparrow\right\rangle\left\langle\uparrow\right| + \left|\downarrow\right\rangle\left\langle\downarrow\right|\right)^{\otimes N}$$

The second order moments are computed as

$$\left\langle J_x^2 \right\rangle_t = \frac{1}{4}\sum_{k,l}\left\langle \sigma_x^{(k)}\sigma_x^{(l)} \right\rangle_t = \frac{N}{4} + \frac{1}{4}\sum_{k\neq l}\underbrace{\cancel{\left\langle \sigma_x^{(k)}\sigma_x^{(l)} \right\rangle_t}}_{=0} = \frac{N}{4}$$

This term is 1 if *k=l*, otherwise it is 0.

$$\left\langle J_x^4 \right\rangle_t = \sum_{k,l,m,n}\left\langle \sigma_x^{(k)}\sigma_x^{(l)}\sigma_x^{(m)}\sigma_x^{(n)} \right\rangle = \frac{3N^2 - 2N}{16}$$

Only those terms are nonzero, for which we have the Pauli spin matrices on an even power.

Moments of the cluster state I.

The moments of J_x, J_y and J_z for a (large enough) cluster state are the same as for the totally mixed state.

How do we compute the moments for a cluster state?

$$\langle J_x^2 \rangle_{cluster} = \frac{1}{4}\sum_{k,l} \langle \sigma_x^{(k)} \sigma_x^{(l)} \rangle_{cluster} = \frac{1}{4}\sum_{k,l} \langle \tilde{\sigma}_x^{(k)} \tilde{\sigma}_x^{(l)} \rangle_{|\uparrow\uparrow\uparrow\uparrow...\rangle_x}$$

$$\tilde{\sigma}_x^{(k)} = U_{PG} \sigma_x^{(k)} U_{PG} = \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)}$$

This term is also 1 if $k=l$, otherwise it is 0 if $N>3$.

$$\langle J_x^2 \rangle_{cluster} = \frac{1}{4}\sum_{k,l} \langle \tilde{\sigma}_x^{(k)} \tilde{\sigma}_x^{(l)} \rangle_{|\uparrow\uparrow\uparrow...\rangle_x} = \frac{N}{4} + \frac{1}{4}\sum_{k\neq l} \langle \tilde{\sigma}_x^{(k)} \tilde{\sigma}_x^{(l)} \rangle_{|\uparrow\uparrow\uparrow...\rangle_x} = \frac{N}{4}$$

Moments of the cluster state II.

However, not all moments of a cluster state are the same as for a totally mixed state. Reason:

$$\left\langle J_x^4 \right\rangle = \sum_{k,l,m,n} \left\langle \sigma_x^{(k)} \sigma_x^{(l)} \sigma_x^{(m)} \sigma_x^{(n)} \right\rangle$$

For N>7 only those terms are nonzero, for which we have the Pauli spin matrices on an even power (as for the totally mixed state). But for N<8 this is not true.

In general, for a cluster state of N qubits, all moments of $J_{x/y/z}$ up to order m<(N+1)/2 are the same, as for the totally mixed state.