

**Optimal spin-squeezing inequalities
for arbitrary spin**
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QCCC, Bernried,
8 October 2011

Outline

1

Motivation

- Why spin squeezing inequalities are important?

2

Multipartite entanglement

3

Quantum experiments with cold gases

- Physical systems
- Collective measurements

4

Spin squeezing

- Squeezing
- Spin squeezing

5

Spin squeezing criteria for $j = 1/2$

- The original criterion
- Generalized criteria for $j = \frac{1}{2}$

6

Spin squeezing inequality for an ensemble of spin- j atoms

- Basic idea for $j > \frac{1}{2}$
- Angular momentum
- SU(d) generators
- Detection of singlets

Why spin squeezing inequalities for $j > \frac{1}{2}$ is important?

- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.
- Most experiments use atoms with $j > \frac{1}{2}$.

Genuine multipartite entanglement

Definition

A state is **(fully) separable** if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \dots \otimes \varrho_N^{(k)}.$$

Definition

A pure multi-qubit quantum state is called **biseparable** if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

Here $|\Psi\rangle$ is an N -qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called **genuine multi-partite entangled**.

k -producibility/ k -entanglement

Definition

A pure state is k -producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle \dots$$

where $|\Phi_j\rangle$ are states of at most k qubits. A mixed state is k -producible, if it is a mixture of k -producible pure states.

[O. Gühne and G. Tóth, *New J. Phys* 2005.]

- In many-particle systems where only collective quantities can be detected, this is the only meaningful characterization of entanglement.
- That is, genuine multipartite entanglement is very difficult to detect in such systems.

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 - Collective measurements
- 4 **Spin squeezing**
 - Squeezing
 - Spin squeezing
- 5 **Spin squeezing criteria for $j = 1/2$**
 - The original criterion
 - Generalized criteria for $j = \frac{1}{2}$
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Physical systems

State-of-the-art in experiments

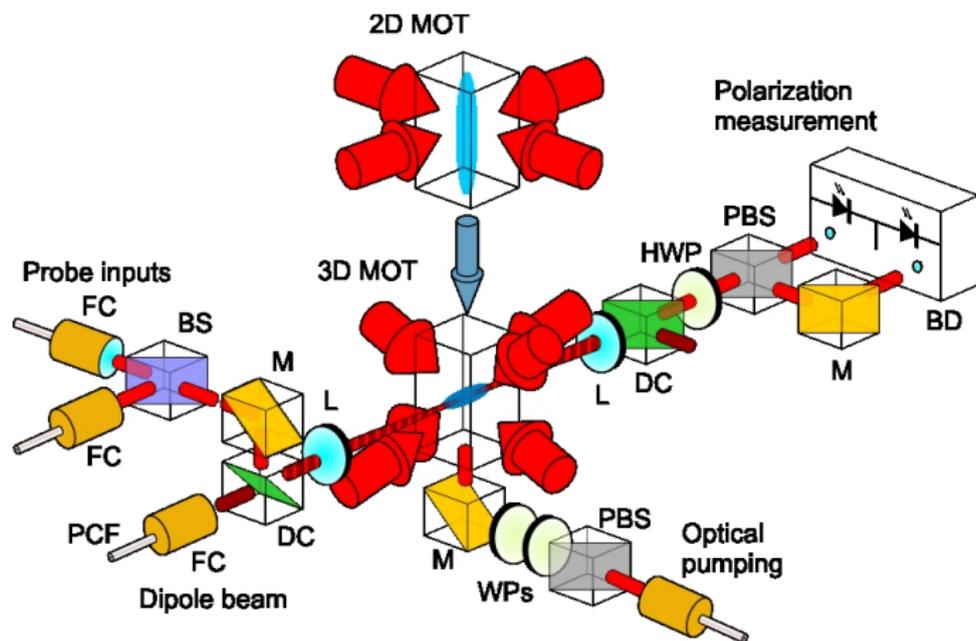
- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with $10^6 - 10^{12}$ atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.

Physical systems II

For example: Spin squeezing in a cold atomic ensemble (not BEC!)



Picture from M.W. Mitchell, ICFO, Barcelona.

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 - Squeezing
 - Spin squeezing
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Many-particle systems for $j=1/2$

- For spin- $\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$$

variances.

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 - Collective measurements
- 4 **Spin squeezing**
 - Squeezing
 - Spin squeezing
- 5 **Spin squeezing criteria for $j = 1/2$**
 - The original criterion
 - Generalized criteria for $j = \frac{1}{2}$
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Squeezing

- The variances of the two quadrature components are bounded

$$(\Delta x)^2(\Delta p)^2 \geq \text{const.}$$

- Coherent states saturate the inequality.
- Squeezed states are the states for which one of the quadrature components have a smaller variance than for a coherent state.
- Can one use similar ideas for spin systems?

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5

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Spin squeezing

- The variances of the angular momentum components are bounded

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z \rangle|^2.$$

If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{|\langle J_z \rangle|}{2}$ then the state is called spin squeezed.

- z is the direction of the mean spin!
- The angular momentum of such a state has a small variance in one direction.
- The variance is large in an orthogonal direction.

[M. Kitagawa and M. Ueda, PRA **47**, 5138 (1993).]

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4

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- Squeezing
- Spin squeezing

5

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The standard spin-squeezing criterion

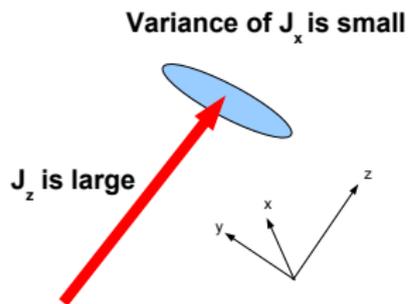
- The spin squeezing criteria for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

If it is violated then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature **409**, 63 (2001).]

- States violating it are like this:



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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$

$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled

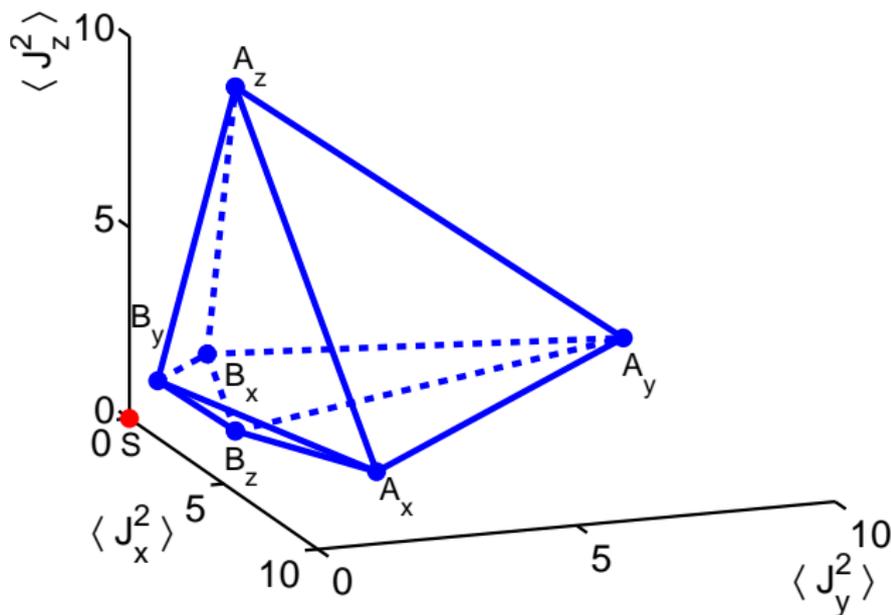
$$\begin{aligned} \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq \frac{N(N+2)}{4}, \\ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \frac{N}{2}, \\ \langle J_k^2 \rangle + \langle J_l^2 \rangle &\leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \\ (N-1) [(\Delta J_k)^2 + (\Delta J_l)^2] &\geq \langle J_m^2 \rangle + \frac{N(N-2)}{4}, \end{aligned}$$

where k, l, m take all the possible permutations of x, y, z .

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
quant-ph/0702219.]

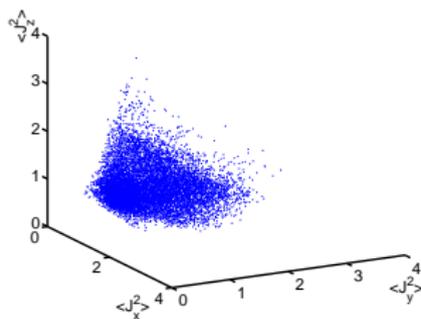
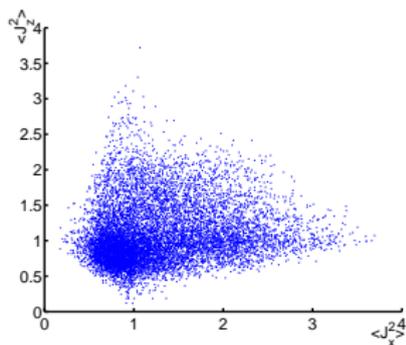
Generalized spin squeezing criteria for $j = \frac{1}{2}$

- The previous inequalities, for fixed $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J_{x/y/z}^2 \rangle$ space. The polytope has six extreme points: $A_{x/y/z}$ and $B_{x/y/z}$.
- For $\langle \vec{J} \rangle = 0$ and $N = 6$ the polytope is the following:



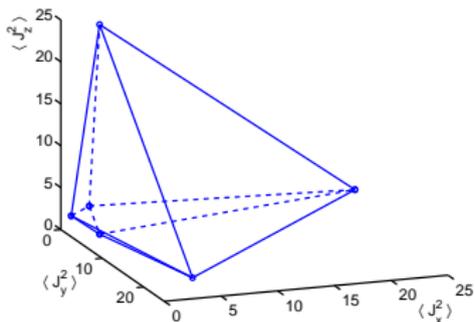
Completeness

- Random separable states:

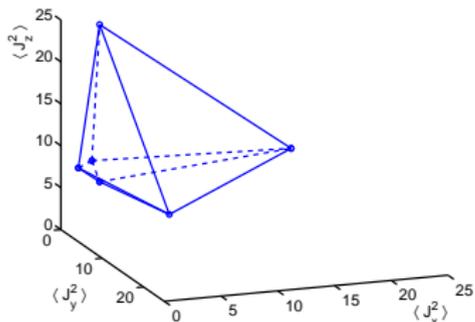


- The completeness can be proved for large N .

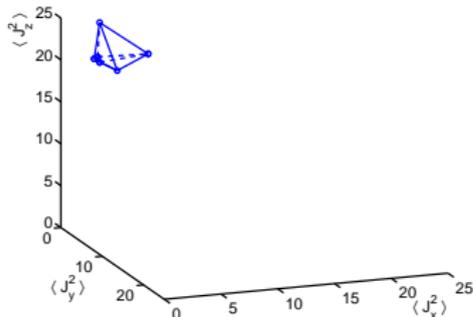
- The polytope for $N = 10$ and $J = (0, 0, 0)$,



$$J = (0, 0, 2.5),$$



and $J = (0, 0, 4.5)$.



Bound entangled states

- The inequalities detect bound entangled states in the thermal states of Heisenberg spin models.
- These states have a positive partial transpose (PPT) with respect to all bipartitions.

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 - Squeezing
 - Spin squeezing
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 - The original criterion
 - Generalized criteria for $j = \frac{1}{2}$
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Basic ideas for the $j > \frac{1}{2}$ case

- Particles with $d > 2$ internal states.
- a_k for $k = 1, 2, \dots, M$ denote single-particle operators with the property $\text{Tr}(a_k a_l) = C \delta_{kl}$, where C is a constant.
- We need the upper bound K for the inequality $\sum_{k=1}^M \langle a_k^{(n)} \rangle^2 \leq K$.
- The N -qudit collective operators used in our criteria will be denoted by

$$A_k = \sum_n a_k^{(n)}.$$

“Modified” quantities for $j > \frac{1}{2}$

- For the $j = \frac{1}{2}$ case, the SSIs were developed based on the first and second moments and variances of the such collective operators.
- For the $j > \frac{1}{2}$ case, we define the modified second moment

$$\langle \tilde{A}_k^2 \rangle := \langle A_k^2 \rangle - \langle \sum_n (a_k^{(n)})^2 \rangle = \sum_{m \neq n} \langle a_k^{(n)} a_k^{(m)} \rangle$$

and the modified variance

$$(\tilde{\Delta} A_k)^2 := (\Delta A_k)^2 - \langle \sum_n (a_k^{(n)})^2 \rangle.$$

- These are essential to get tight equations for $j > \frac{1}{2}$.

Basic equation

- For separable states, i.e., for states that can be written as a mixture of product states,

$$(N - 1) \sum_{k \in I} (\tilde{\Delta} A_k)^2 - \sum_{k \notin I} \langle (\tilde{A}_k)^2 \rangle \geq -N(N - 1)K$$

holds, where each index set $I \subseteq \{1, 2, \dots, M\}$ defines one of the 2^M inequalities.

- Note that $I = \emptyset$ and $I = \{1, 2, \dots, M\}$ are among the possibilities.

Derivation

- We consider product states of the form $|\Phi\rangle = \otimes_n |\phi_n\rangle$. For such states, we have $(\Delta\tilde{A}_k)^2_\Phi = -\sum_n \langle a_k^{(n)} \rangle^2$.
- Hence, the left-hand side of the inequality equals

$$\begin{aligned} & -\sum_n (N-1) \sum_{k \in I} \langle a_k^{(n)} \rangle^2 - \sum_{k \notin I} \left(\langle A_k \rangle^2 - \sum_n \langle a_k^{(n)} \rangle^2 \right) \\ & \geq -\sum_n (N-1) \sum_{k=1}^M \langle a_k^{(n)} \rangle^2 \geq -N(N-1)K \end{aligned}$$

- We used that $\langle A_k \rangle^2 \leq N \sum_n \langle a_k^{(n)} \rangle^2$.
- The equation is saturated by all states of the form $|\phi\rangle^{\otimes N}$.

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 - Spin squeezing
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The inequalities for $j > \frac{1}{2}$ with the angular momentum coordinates

- Application 1:

$$a_k = \{j_x, j_y, j_z\}.$$

- For spin- j particles for $j > \frac{1}{2}$, we can measure the collective angular momentum operators:

$$J_l := \sum_{k=1}^N j_l^{(k)},$$

where $l = x, y, z$ and $j_l^{(k)}$ are the angular momentum coordinates [i.e., SU(2) generators].

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$$

variances.

The inequalities for $j > \frac{1}{2}$ with the angular momentum coordinates II

- For fully separable states of spin- j particles, all the following inequalities are fulfilled

$$\begin{aligned}\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq Nj(Nj + 1), \\ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq Nj, \\ \langle \tilde{J}_k^2 \rangle + \langle \tilde{J}_l^2 \rangle - N(N-1)j^2 &\leq (N-1)(\tilde{\Delta} J_m)^2, \\ (N-1) [(\tilde{\Delta} J_k)^2 + (\tilde{\Delta} J_l)^2] &\geq \langle \tilde{J}_m^2 \rangle - N(N-1)j^2,\end{aligned}$$

where k, l, m take all possible permutations of x, y, z .

- Violation of any of the inequalities implies entanglement.

Completeness

- In the large N limit, the inequalities with the angular momentum are **complete**.
- That is, it is not possible to come up with a new entanglement conditions with based on $\langle J_k \rangle$ and seed on $\langle J_k \rangle$ and $\langle \tilde{J}_k^2 \rangle$ that detect states not detected by these inequalities.

Mapping qubit inequalities to qudits

- Take an inequality valid for N -qubit separable states of the form

$$f(\{\langle \mathbf{J}_l \rangle\}_{l=x,y,z}, \{\langle \tilde{\mathbf{J}}_l^2 \rangle\}_{l=x,y,z}) \geq \text{const.},$$

where f is a concave function. All of the generalized SSIs have this form.

- An entanglement condition can be transformed to a criterion for a system of N spin- j particles by the substitution

$$\langle \mathbf{J}_l \rangle \rightarrow \frac{1}{2j} \langle \mathbf{J}_l \rangle, \quad \langle \tilde{\mathbf{J}}_l^2 \rangle \rightarrow \frac{1}{4j^2} (\langle \tilde{\mathbf{J}}_l^2 \rangle).$$

The usual spin squeezing inequality for $j > \frac{1}{2}$

- The standard spin-squeezing inequality becomes

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} + \frac{\sum_n (j^2 - \langle (j_x^{(n)})^2 \rangle)}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

This inequality is violated only if there is entanglement between the spin- j particles.

- Due to the second, nonnegative term on the left-hand side, for $j > \frac{1}{2}$ there are states that violate the original inequality, but do not violate this one. **Thus, there is spin squeezing without entanglement between the particles.**
- Our spin squeezing inequalities are strictly stronger than the original inequality.

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 - Squeezing
 - Spin squeezing
- 5 **Spin squeezing criteria for $j = 1/2$**
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 - Generalized criteria for $j = \frac{1}{2}$
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The inequalities for $j > \frac{1}{2}$ with the G_k 's

- Application 2:

$$a_k = SU(d) \text{ generators.}$$

- For spin- j particles for $j > 1/2$, we can measure the collective operators:

$$G_l := \sum_{k=1}^N g_l^{(k)},$$

where $l = 1, 2, \dots, d^2 - 1$ and $g_l^{(k)}$ are the $SU(d)$ generators.

- We can also measure the

$$(\Delta G_l)^2 := \langle G_l^2 \rangle - \langle G_l \rangle^2$$

variances.

The inequalities for $j > \frac{1}{2}$ with the G_k 's

- For a system of d -dimensional particles, we can define collective operators based on the SU(d) generators $\{g_k\}_{k=1}^M$ with $M = d^2 - 1$ as $G_k = \sum_{n=1}^N g_k^{(n)}$.
- The SSIs for G_k have the general form

$$(N-1) \sum_{k \in I} (\tilde{\Delta} G_k)^2 - \sum_{k \notin I} \langle (\tilde{G}_k)^2 \rangle \geq -2N(N-1) \frac{(d-1)}{d}.$$

For instance, for the $d = 3$ case, the SU(d) generators can be the eight Gell-Mann matrices.

- I is a subset of indices between 1 and M . We have 2^M equations!

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One of the generalized spin squeezing criteria

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

[G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth,
Optimal spin squeezing inequalities for arbitrary spin,
arXiv:1104.3147.]

Maximally violating states

- For $N = d$, the multipartite singlet state maximally violates the condition with $\sum_k (\Delta G_k)^2 = 0$.
- For $N < d$, there is no quantum states for which $\sum_k (\Delta G_k)^2 = 0$.
- This can be seen as follows. It is not possible to create a completely antisymmetric state of d -state particles with less than d particles.

The criterion

A condition for two-producibility for N qudits of dimension d is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-2).$$

A condition for separability is

$$\sum_k (\Delta G_k)^2 \geq 2N(d-1).$$

Noise tolerance

- Let us consider SU(d) singlet states (i.e., states with $\langle G_k^2 \rangle = 0$) mixed with white noise as

$$\rho_{\text{noisy}} = (1 - p_{\text{noise}})\rho_{\text{singlet}} + p_{\text{noise}} \frac{1}{d^N} \mathbb{1}.$$

- Direct calculation shows that such a state is detected as entangled if

$$p_{\text{noise}} < \frac{d}{d+1}.$$

Thus, the noise tolerance in detecting SU(d) singlets is increasing with d !

Advantages of criteria for $j > \frac{1}{2}$

- Most atoms have $j > \frac{1}{2}$. No need to create spin-1/2 subsystems artificially
- Manipulation is possible with magnetic fields rather than with lasers.
- New experiments can be proposed.

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Iñigo Urizar-Lanz	2nd year Ph.D. Student
Giuseppe Vitagliano	2nd year Ph.D. Student

- Topics

- Multipartite entanglement and its detection
- Metrology, cold gases
- Collaborating on experiments:
 - Weinfurter group, Munich, (photons)
 - Mitchell group, Barcelona, (cold gases)

- Funding:

- European Research Council starting grant 2011-2016, 1.3 million euros
- CHIST-ERA QUASAR collaborative EU project
- Grants of the Spanish Government and the Basque Government

Summary

- We presented a full set of generalized spin squeezing inequalities with the angular momentum coordinates for $j > 1/2$.
- We presented a large set of inequalities with the other collective operators that can be measured.
- These might make possible new experiments and make existing experiments simpler.

See: G. Vitagliano, P. Hyllus, I.L. Egusquiza, and G. Tóth,
Optimal spin squeezing inequalities for arbitrary spin, arXiv:1104.3147.

THANK YOU FOR YOUR ATTENTION!

