Extremal properties of the variance and the quantum Fisher information; Phys. Rev. A 87, 032324 (2013).

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Outline

1. **Motivation**
   - Why variance and the quantum Fisher information are important?
2. **Variance and quantum Fisher information**
   - Basic definitions
   - Entanglement detection with the variance
   - Entanglement detection with the quantum Fisher information
3. **Generalized variance and quantum Fisher information**
   - Generalized variance
   - Generalized quantum Fisher information
   - Generalized quantities in the literature

Why the variance and the quantum Fisher information are important?

- Variance appears in all areas of physics.
- Quantum Fisher information is a central notion in metrology.
- Concave roofs, convex roofs are interesting in entanglement theory.

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Variance

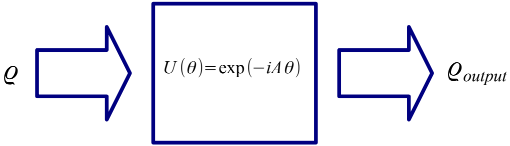
- The variance is defined as

$$(\Delta A)^2{}_{\varrho} = \langle A^2 \rangle_{\varrho} - \langle A \rangle_{\varrho}^2.$$

- The variance is concave.

Quantum Fisher information (QFI)

- The parameter θ must be estimated by measuring he output state :



- Cramér-Rao bound

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q^{\text{usual}}[\varrho, A]}}.$$

- The quantum Fisher information is

$$F_Q^{\text{usual}}[\varrho, A] = 2\sum_{i,j} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |A_{ij}|^2.$$

- For pure states, $F_Q^{\text{usual}}[\varrho, A] = 4(\Delta A)^2{}_{\varrho}$, and it is convex.

[E.g., P. Hyllus, O. Gühne, and A. Smerzi, Phys. Rev. A 82, 012337 (2010).]

Outline

Entanglement detection with the variance

- Two properties of the variance are used:
  - For pure states, it is $\langle A^2\rangle_\Psi - \langle A\rangle_\Psi^2$.
  - It is concave.
- Any other quantity with these propeties could be used instead of the variance.
- If it were smaller than the variance, then it would even be better than the variance for this purpose.

[O. Gühne, Phys. Rev. Lett. 92, 117903 (2004).]

Outline

Entanglement detection with the QFI

- Two properties of the QFI are used:
  - For pure states, it is $4(\langle A^2\rangle_\Psi - \langle A\rangle_\Psi^2)$.
  - It is convex.
- Any other quantity with these propeties could be used instead of the QFI.
- If it were larger than the usual quantum Fisher information, then it would even be better for this purpose.

[L. Pezze and A. Smerzi , Phys. Rev. Lett. 102, 100401 (2009).]

Outline

Generalized variance

Definition 1. Generalized variance $\mathrm{var}_\varrho(A)$ is defined as follows.

1. For pure states, we have
$$\mathrm{var}_\Psi(A) = (\Delta A)^2{}_\Psi.$$
2. For mixed states, $\mathrm{var}_\varrho(A)$ is concave in the state.

Definition 2. The minimal generalized variance $\mathrm{var}_\varrho^{\min}(A)$ is defined as follows.

1. For pure states, it equals the usual variance
$$\mathrm{var}_\Psi^{\min}(A) = (\Delta A)^2{}_\Psi,$$
2. For mixed states, it is defined through a concave roof construction
$$\mathrm{var}_\varrho^{\min}(A) = \sup_{\{p_k,\Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k},$$
where
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

Theorem 1

Theorem 1.

The minimal generalized variance is the usual variance

$$\mathrm{var}_{\varrho}^{\min}(A) = (\Delta A)^2_{\varrho}.$$

In other words, the variance its own concave roof.

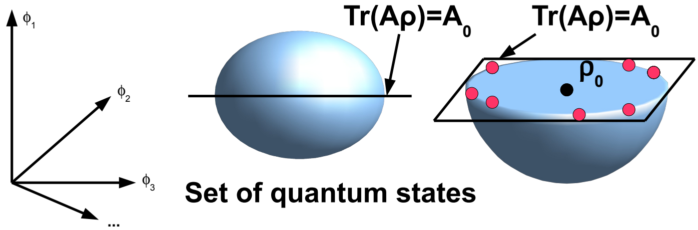
Hand waving proof:

$$(\Delta A)^2_{\varrho} = \sum_k p_k (\Delta A)^2_{\Psi_k} + (\langle A \rangle_{\Psi_k} - \langle A \rangle_{\varrho})^2.$$

You can always find a decomposition such that $\langle A \rangle_{\Psi_k} = \langle A \rangle_{\varrho}$ for all k.

Theorem 1

Hand waving proof, continuation; geometric argument:



For details, please see arxiv:1109.2831.

Outline

Generalized quantum Fisher information

Definition 3. Generalized quantum Fisher information $F_Q[\varrho, A]$:

1. For pure states, we have

$$F_Q[\varrho, A] = 4(\Delta A)^2{}_{\Psi}.$$

   The factor 4 appears for historical reasons.

2. For mixed states, $F_Q[\varrho, A]$ is convex in the state.

Definition 4. Maximal quantum Fisher information $F_Q^{\max}[\varrho, A]$:

1. For pure states, it equals four times the usual variance

$$F_Q^{\max}[\varrho, A] = 4(\Delta A)^2{}_{\Psi}.$$

2. For mixed states, it is defined through a convex roof construction

$$F_Q^{\max}[\varrho, A] = 4 \inf_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k}.$$

Theorem 2

Theorem 2.

For rank-2 states

$$F_Q^{\max}[\varrho, A] = F_Q^{\mathrm{usual}}[\varrho, A].$$

For an analytic proof, see G. Tóth and D. Petz, arxiv:1109.2831.

In other words, the quantum Fisher information is four times the convex roof of the variance for rank-2 states.

Numerics for rank>2

- The maximal generalized q. Fisher information can be written as

$$F_Q^{\max}[\varrho, A] = 4\left(\langle A^2\rangle_\varrho - \sup_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k \langle A\rangle_{\Psi_k}^2\right).$$

- Rewriting the term quadratic in expectation values as an operator acting on a bipartite system

$$F_Q^{\max}[\varrho, A] = 4\left(\langle A^2\rangle_\varrho - \sup_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k \langle A \otimes A\rangle_{\Psi_k \otimes \Psi_k}\right).$$

- Further transformations lead to

$$F_Q^{\max}[\varrho, A] = 4\left(\langle A^2\rangle_\varrho - \sup_{\{p_k, |\Psi_k\rangle\}} \langle A \otimes A\rangle_{\sum_k p_k |\Psi_k\rangle\langle\Psi_k|^{\otimes 2}}\right).$$

Numerics for rank>2 II

- Hence we obtain that

$$F_Q^{\max}[\varrho, A]. = 4\left(\langle A^2\rangle_\varrho - \sup_{\substack{\varrho_{\rm ss}\in S_{\rm s},\\ {\rm Tr}_1(\varrho_{\rm ss})=\varrho}} \langle A\otimes A\rangle_{\varrho_{\rm ss}}\right),$$

where S_s are symmetric separable states.

- Instead of the separable states, we can do the optimization for PPT states or states with a PPT symmetric extension.

- Extensive numerics on random ϱ and A confirm that

$$F_Q = F_Q^{max}$$

holds within a large degree of accuracy.

Outline

Generalized variance and quantum Fisher information in the literature

- Generalized variances and quantum Fisher informations of D. Petz.
- Defines a variance and a corresponding quantum Fisher information for each standard matrix monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$.
- *Surprisingly*, his variances and quantum Fisher information definitions fit the definitions of this presentation.
- Our quantities are extremal even within the sets defined by Petz et al. However, our definitions are broader.

D. Petz, *Quantum Information Theory and Quantum Statistics* (Springer, 2008).

D. Petz, J. Phys. A: Math. Gen. **35**, 79 (2003).

P. Gibilisco, F. Hiai and D. Petz, IEEE Trans. Inform. Theory **55**, 439 (2009).

F. Hiai and D. Petz, From quasi-entropy, http://arxiv.org/abs/1009.2679.

Conjecture

Conjecture

We conjecture that

$$F_Q = F_Q^{max}$$

for density matrices of any rank and for any Hermitian A.

Conjecture based on

- Analytics for rank 2
- Extensive numerics for rank>2
- Statement is true for a large subset

Follow-up

- Proof: Sixia Yu, arxiv 1302.5311.
- We should look for connections to
  [B.M. Escher,R.L.de Matos Filho, and L. Davidovich, Nature Phys. (2011)] .

Summary of the two relations

- Two *sharp* inequalities

$$\frac{1}{4}F_Q[\varrho, A] \le \sum_k p_k (\Delta A)^2_k \le (\Delta A)^2.$$

Summary

- We defined the generalized variance and the generalized quantum Fisher information.
- We found that the variance is its own concave roof, while the quantum Fisher information is its own convex roof.

See:
G. Tóth and D. Petz,
Extremal properties of the variance and the quantum Fisher information, Phys. Rev. A 87, 032324 (2013).
arxiv:1109.2831.

THANK YOU FOR YOUR ATTENTION!

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