Efficient algorithm for multi-qudit twirling for ensemble quantum computation Phys. Rev. A 2007; quant-ph/0609052

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Outline

1. Motivation
2. Usual algorithm for multi-qudit twirling
3. Twirling with a recursive algorithm
4. Realization on a quantum computer
5. Case of imperfect random unitaries and deterministic twirling
6. Connections to numerical integration over $U(d)$ and $SU(d)$ and relations to other fields

Outline

Motivation

- Bipartite twirling plays an important role in entanglement purification protocols.
- Can be used for entanglement detection with few measurements:
  - The twirled state which can be described by much fewer parameters than the original one.
  - Twirling does not increase entanglement.
- Integrals over the unitary group appear in other areas of physics. An algorithm for twirling, which can efficiently be used on a digital computer, can also be useful for computing such integrals.

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Twirling by averaging over randomly rotated matrices

- For a given density matrix ρ the twirled state is defined as

$$\mathbf{P}\rho := \int_{U\in U(d)} U^{\otimes N}\rho(U^{\otimes N})^{\dagger} dU,$$

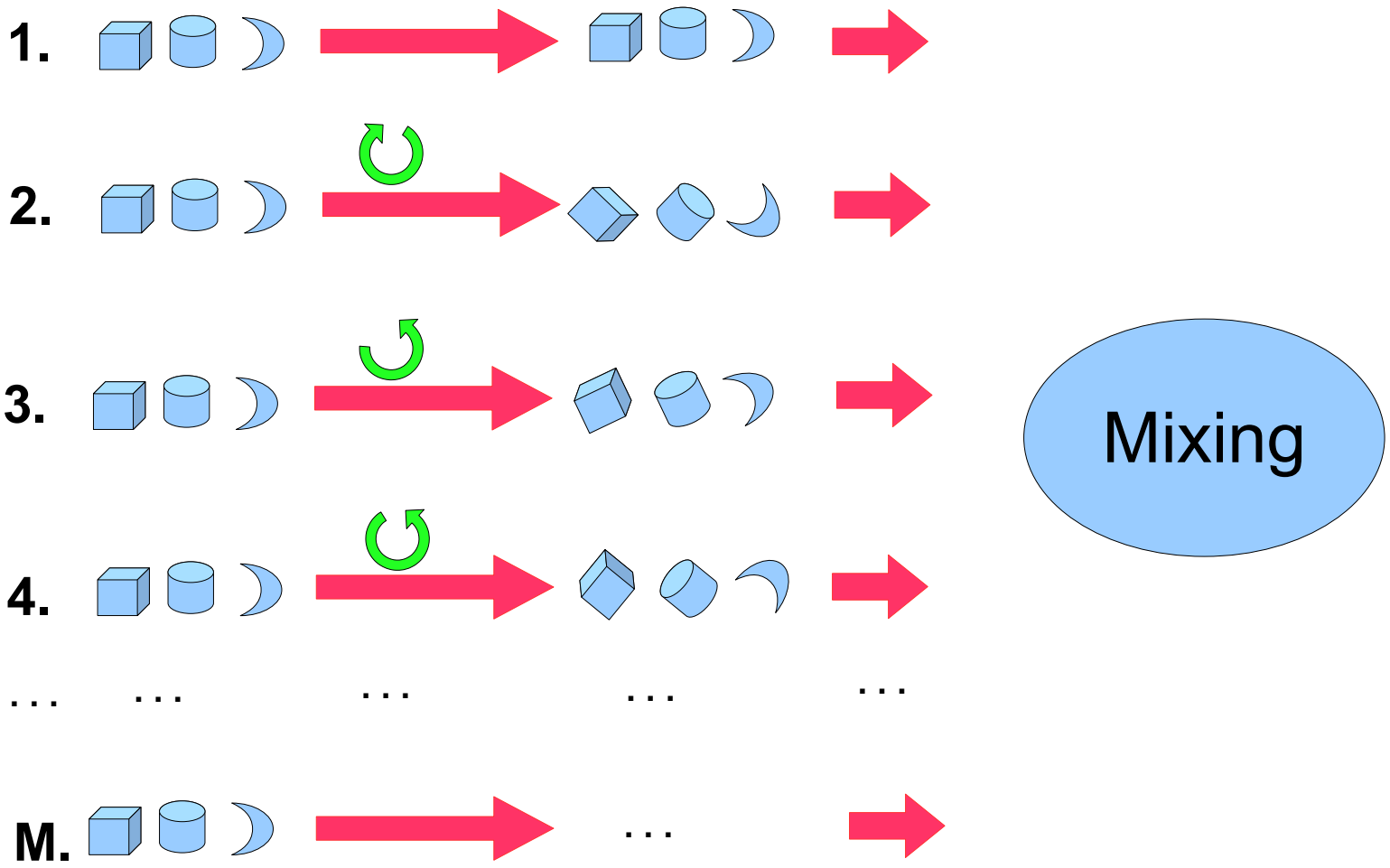
where $U(d)$ is the group of d-dimensional unitary matrices, N is the number of qudits, and dU is the normalized Haar measure over $U(d)$.

- $\mathbf{P}\rho$ can be approximated by an average of a finite number of randomly rotated density matrices

$$\mathbf{P}_M\rho := \frac{1}{M}\left[\rho + \sum_{k=1}^{M-1} U_k^{\otimes N}\rho(U_k^{\otimes N})^{\dagger}\right].$$

Here M denotes the number of terms and we assume that the unitaries $\{U_k\}$ are distributed uniformly in $U(d)$ according to the Haar measure.

Twirling by averaging over randomly rotated matrices II



Convergence of the usual method

- To analyze the error, we introduce an expectation value or average over the different choices for U_k as

$$\langle A \rangle := \int A dU_1 dU_2 dU_3 ...$$

- Simple calculations show that the average squared error of a particular initial state, ρ, decreases algebraically as M^{-1}

$$\langle ||\mathbf{P}_M \rho - \mathbf{P}\rho||^2 \rangle = \frac{1}{M} \left(||\rho||^2 - ||\mathbf{P}\rho||^2 \right),$$

where $||A||^2 := \mathrm{Tr}(A^\dagger A)$ is the Hilbert-Schmidt norm.

Problem with the usual method

- Individual addressing of systems is needed.
- Twirling is realized in practice as temporal averaging. Thus, the execution time is proportional to the number of systems in the ensemble. What if the there are many systems in the ensemble?
- The error decreases slowly (polynomially) with the number of elementary steps.

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Our proposal for efficient twirling

- Let us now consider repeated applications of $\mathbf{P}_2$. Remember:

$$\mathbf{P}_2\rho := \frac{1}{2}\left[\rho + U^{\otimes N}\rho(U^{\otimes N})^\dagger\right],$$

where U is random. After M iterations, the outcome is

$$\mathbf{Q}_M\rho := \mathbf{P}_2\mathbf{P}_2...\mathbf{P}_2\rho = \left(\prod_{k=1}^{M}\mathbf{P}_2\right)\rho.$$

- The error decreases exponentially with M

$$\langle||\mathbf{Q}_M\rho - \mathbf{P}\rho||^2\rangle = \left(||\rho||^2 - ||\mathbf{P}\rho||^2\right)2^{-M}.$$

- The error can also be computed for the *superoperator*. It also decays as $\propto 2^{-M}$. The advantage of this approach is that it gives statements independent from a concrete density matrix.

Superoperators

- Density matrices are vectors in a Hilbert space. Thus it is convenient to switch from matrix notation

$$\rho = \sum_{kl} \rho_{kl} |k\rangle\langle l|$$

and treat the matrices as vectors defined by

$$v_\rho = \sum_{kl} \rho_{kl} |l\rangle \otimes |k\rangle.$$

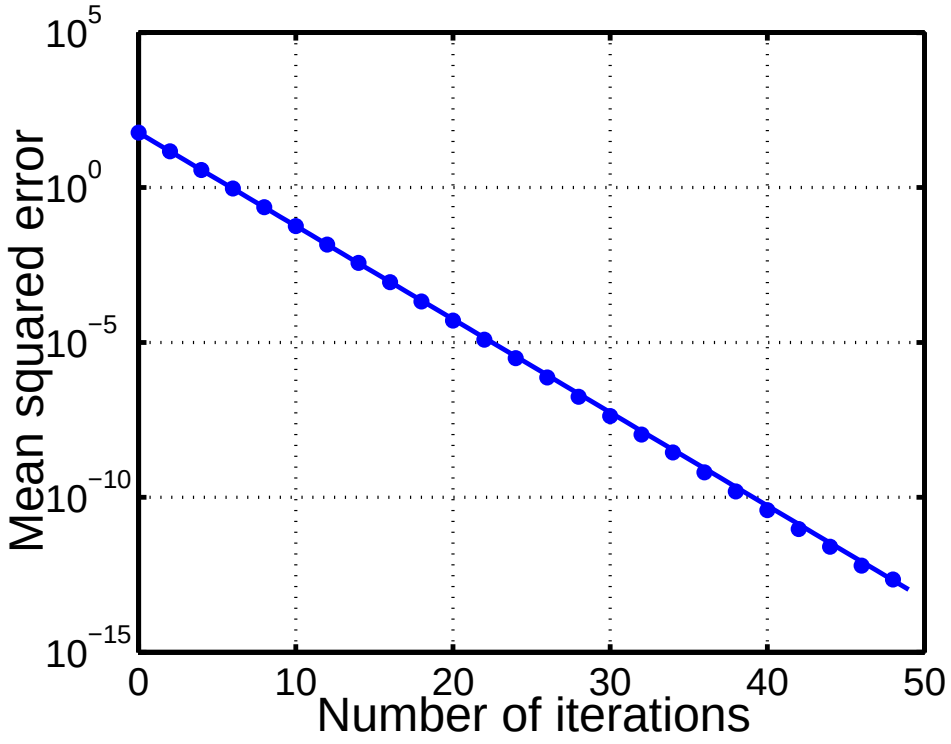
- Any physically allowed transformation of the density matrix is a linear positive map and it can be written as a matrix acting on v_ρ

$$v_{\rho'} = S v_\rho.$$

- The distance between superoperators can be measured in the form of Hilbert-Schmidt norm of their difference

$$\|S - \tilde{S}\|^2 := \mathrm{Tr}[(S - \tilde{S})(S - \tilde{S})^\dagger].$$

Simple simulation example



Mean squared error for the recursive method with random matrices when applied on three-qubit states. We plot the average over 10000 realizations (dotted) and the theoretical prediction computed (solid line).

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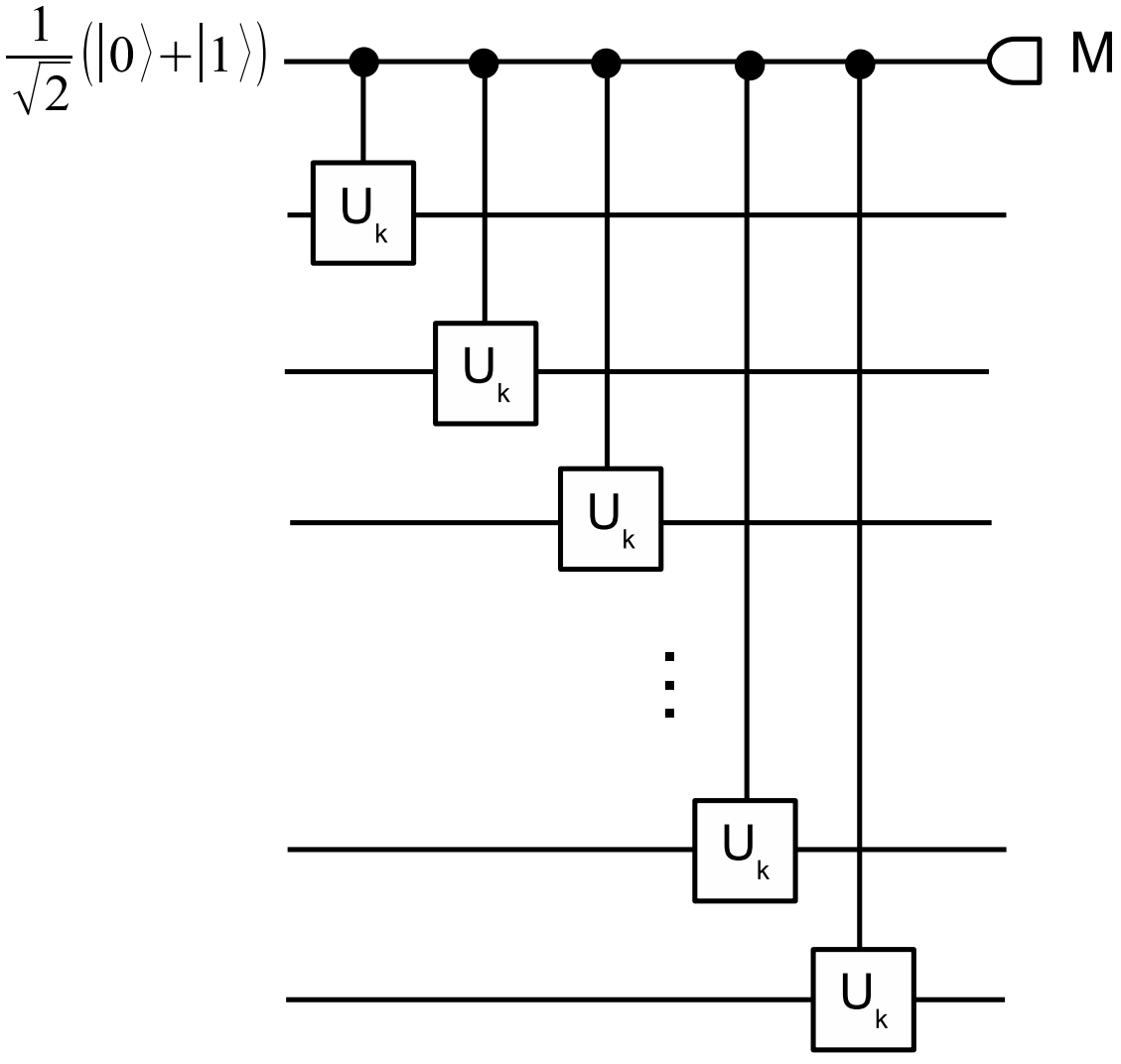
Quantum circuit for a single iteration step

The implementation does not need an individual access to the spins. Thus it is suitable for ensemble quantum computing.



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Imperfect random unitaries

- An imperfect random unitary generator can be characterized by the distribution $f(U)$ describing the probability density for getting U

$$f(U) := p_g g(U) + (1 - p_g).$$

- The error decays as

$$\propto \left(\frac{2}{1 + p_g^2} \right)^{-M}.$$

  Thus we have convergence if $p_g < 1$.

- In other words, our algorithm still converges to the twirled state and the error decays exponentially if

$$\inf_U f(U) > 0.$$

Deterministic twirling

- What happens if unitaries are not random but they are chosen deterministically from a small set such that they are cyclically alternating.

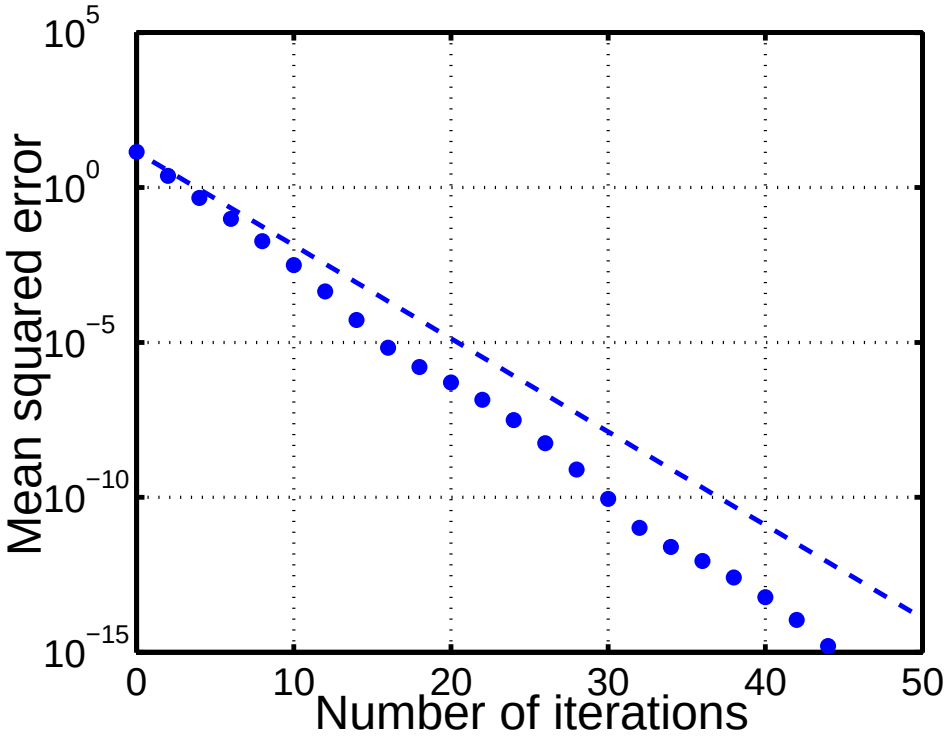
- Let us consider the two-qubit case and choose the two unitaries as

$$\begin{aligned} U_x &:= e^{ic\sigma_x}, \\ U_z &:= e^{ic\sigma_z}, \end{aligned}$$

  where $\sigma_{x/z}$ are Pauli spin matrices and $c = 1.0894$.

- Numerical calculations with the superoperator show that we again have exponential convergence.

Deterministic twirling II



Time dependence of the error for two qubits for the deterministic method using two unitaries.

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Numerical integration over $U(d)$ and $SU(d)$

- Our approach can straightforwardly be generalized for integrating expressions of the type

$$I := \int_{U\in U(d)} \mathrm{Tr}(A_1 U)\mathrm{Tr}(A_2 U)...\mathrm{Tr}(A_m U) \\ \mathrm{Tr}(B_1 U^\dagger)\mathrm{Tr}(B_2 U^\dagger)...\mathrm{Tr}(B_n U^\dagger) dU,$$

  where A_k and B_k are $d \times d$ matrices.

- These ideas seem to work also when integrating over a subgroup of $U(d)$, in particular, over the special unitary group $SU(d)$. Such integrals appear, for example, in quantum chromodynamics.

Other integrals that we can compute

- Integrals containing the power of U:

$$\int_{U\in U(d)} dU \mathrm{Tr}(AU^{\alpha})\mathrm{Tr}(BU^{\beta})...$$

- Integrals containing products of U_k's, for example,

$$\int_{U_1,U_2,...\in U(d)} dU_1 dU_2 ... \mathrm{Tr}(AU_1U_2)\mathrm{Tr}(BU_2U_3)...$$

Relation to other fields: Unitary t-designs

- Unitary t-designs:

$$\int_{U\in U(d)} f(U,U^\dagger)dU = \frac{1}{D}\sum_{k=1}^{D} f(U_k,U^\dagger).$$

  The integral is replaced by a finite sum. For certain class of $f(U,U^\dagger)$ a particular choice of $\{U_k\}$ gives exact equality.
- In particular, we talk about a t-design, if $f(U,U^\dagger)$ is a polynomial of the elements of U and $U^\dagger$ of degree t.
- If we have a t-design, we can use it for twirling $N = t$ qudits.
- Twirling for qubits for $N = 1$:

$$\int_{U\in U(d)} U\rho U^\dagger dU = \frac{1}{4}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z).$$

- Twirling for qubits for $N = 2$: 12 unitaries are enough.

[E.g., D. Gross and J. Eisert Journal of Mathematical Physics, 2007; C Dankert, R Cleve, J Emerson, E Livine, quant-ph/0606161]

Relation to other fields: Expander maps

- Let us consider the map

$$\rho \mapsto \frac{1}{D} \sum_{k=1}^{D} U_k \rho U_k^\dagger,$$

with random U_k's.

- For $D = \infty$ this brings every density matrix into the completely mixed matrix. Thus the superoperator has a single eigenvalue 1, the other eigenvalues are 0.

- For finite D, the corresponding superoperator has a single eigenvalue 1, while the other eigenvalues are smaller than 1.

[M. Hastings, arXiv:0706.0556, also in Phys. Rev. A; A. Harrow, arXiv:0709.1142;

A. Ben-Aroya, A. Tha-Shma, arXiv:quant-ph/0702129; D. Gross, J. Eisert, arXiv: 0710.0651]

Conclusions

- We showed how to realize twirling on a quantum computer or on a digital computer efficiently.
- We presented an iterative method which uses a random unitary at each step
- The error compared to perfect twirling decays exponentially.
- The method works also with an imperfect random source or with deterministically chosen unitaries.
- It can be realized with a simple quantum circuit which does not need an individual access to the qudits.

For further details please see the following links:

G. Tóth and J.J. García-Ripoll, Phys. Rev. A 75, 042311 (2007).

http://www.gtoth.hu/

*** THANK YOU ***