Multipartite entanglement and its experimental detection

Géza Tóth

[1]Theoretical Physics, The University of the Basque Country, Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Research Institute for Solid State Physics and Optics, Budapest, Hungary

Tihany, August 31, 2010

Outline

Why is multipartite entanglement interesting?

- There have been many experiments recently aiming to create many-body entangled states.
- Quantum Information Science can help to find good targets for such experiments.
- Multipartite entangled states are needed in applications such as metrology.

Outline

Two qubits

Fact

Remember: There is only a single type of two-qubit entanglement.

- From a single copy of any pure entangled two-qubit state, we can get to any other entangled two-qubit state through Stochastic Local Operations and Classical Communication (SLOCC).

  That is, for any entangled $|\Psi\rangle$ and $|\Phi\rangle$, there are invertible A and B such that

  $$|\Psi\rangle = A \otimes B|\Phi\rangle.$$

  Note that A and B do not have to be Hermitian.

Bipartite systems

- For the mixed case, the definition of a separable state is (Werner 1989)

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}.$$

Definition

Local Operation and Classical Communications (LOCC):

- Single-party unitaries,
- Single party von Neumann measurements,
- Single party POVM measurements,
- We are even allowed to carry out measurement on party 1 and depending on the result, perform some other operation on party 2 ("Classical Communication").

LOCC and entanglement

It is not possible to create entangled states from separable states, with LOCC.

Distillation

- From many entangled particle pairs we want to create fewer strongly entangled pairs with LOCC.

Two qubits - mixed states

Fact

Remember: There is only a single type of two-qubit entanglement.

- From many copies of mixed entangled states, we can always distill a singlet using Local Operations and Classical Communication (LOCC).

The positivity of the partial transpose (PPT) criterion

Definition

For a separable state ϱ, the partial transpose is always positive semidefinite

$$\varrho^{T1} \geq 0.$$

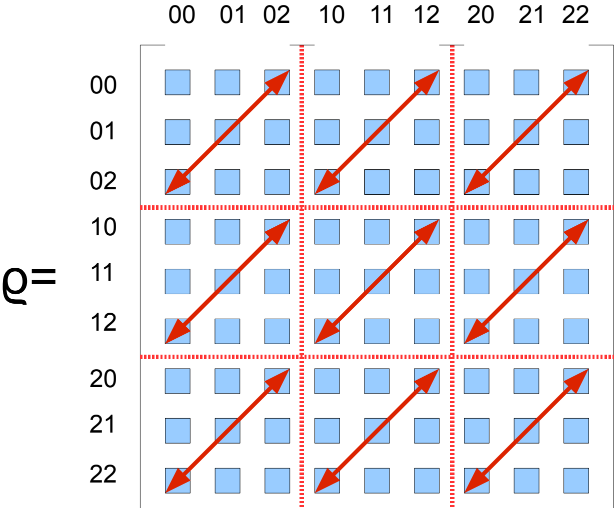
If a state does not have a positive semidefinite partial transpose, then it is entangled. [A. Peres, PRL 1996; Horodecki *et al.*, PLA 1997.]

- Partial transpose means transposing according to one of the two subsystems.
- For separable states

$$(T \otimes \mathbb{1})\varrho = \varrho^{T1} = \sum_k p_k (\varrho_k^{(1)})^T \otimes \varrho_k^{(2)} \geq 0.$$

The positivity of the partial transpose (PPT) criterion II

- How to obtain the partial transpose of a general density matrix? Example: 3×3 case.



Measuring entanglement, bipartite case

- Entanglement of formation:
  - Pure states: Entropy of the reduced state
  - Mixed states: Defined by a convex roof construction

$$E_F(\varrho) = \min_{\{|\Psi_k\rangle, p_k\}: \varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|} \sum_k p_k E_F(|\Psi_k\rangle).$$

- Negativity: = (-1) times the sum of the negative eigenvalues of the partial transpose. (Vidal, Werner)

Three-qubit pure states

- $|\Psi\rangle$ and $|\Phi\rangle$ are equivalent under SLOCC if there are invertible A, B and C such that
$$|\Psi\rangle = A \otimes B \otimes C|\Phi\rangle.$$

☛ **Talk by PÉTER LÉVAY**

Three-qubit mixed states

Six classes:

Class #1: fully separable states $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \varrho_3^{(k)}$

Class #2: (1)(23) biseparable states $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_{23}^{(k)}$, not in Class #1

Class #3: (12)(3) biseparable states $\sum_k p_k \varrho_{12}^{(k)} \otimes \varrho_3^{(k)}$, not in Class #1

Class #4: (13)(2) biseparable states $\sum_k p_k \varrho_{13}^{(k)} \otimes \varrho_2^{(k)}$, not in Class #1

Class #5: W-class states:
mxtr of pure (W ∪ Bisep ∪ Sep)-class states, not in Classes #1-4

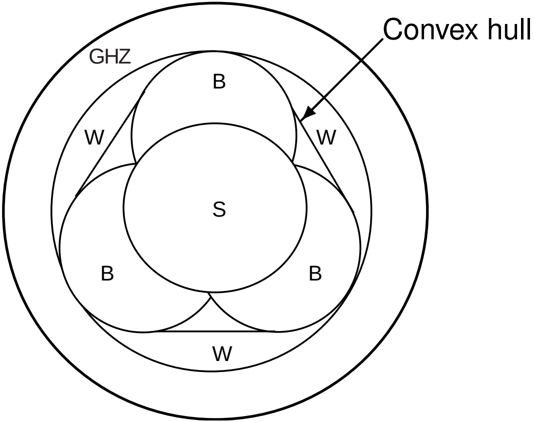
Class #6: GHZ-class states: mxtr of pure (GHZ ∪ W ∪ Bisep ∪ Sep)-class states, not in Classes #1-5

Biseparable states: mixture of states of classes #2, #3 and #4.

Three-qubit mixed states II

- The extension of the classification of pure states to mixed states leads to convex sets:

A. Acín, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001)



Witnesses for GHZ and W-class states

Entanglement witnesses for detecting states of a given class:

GHZ-class states

$$\mathcal{W}_{\mathrm{GHZ}}^{(P)} := \frac{3}{4}\mathbb{1} - |GHZ\rangle\langle GHZ|.$$

W-class states

$$\mathcal{W}_{\mathrm{W}}^{(P)} := \frac{2}{3}\mathbb{1} - |W\rangle\langle W|.$$

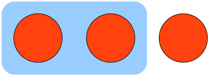
$$\mathcal{W}_{\mathrm{GHZ}}^{(P)} := \frac{1}{2}\mathbb{1} - |GHZ\rangle\langle GHZ|.$$

A. Acín, D. Bruss, M. Lewenstein, A. Sanpera, Phys. Rev. Lett. 87, 040401 (2001)

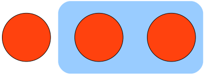
States that are biseparable with respect to all bipartitions

- There are states that are biseparable with respect to all the three bipartitions, but they are *not* fully separable.

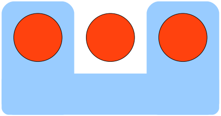
$$\varrho = \sum_k p_k \varrho_{12}^{(k)} \otimes \varrho_3^{(k)}$$



$$\varrho = \sum_k p'_k \varrho_{1}^{(k)} \otimes \varrho_{23}^{(k)}$$



$$\varrho = F_{12} \sum_k p''_k \varrho_{2}^{(k)} \otimes \varrho_{13}^{(k)} F_{12}$$



W. Dür, J.I. Cirac, Phys. Rev. A 61, 042314 (2000)

Outline

More than three qubits

- 4 qubits: There are 9 families and infinite number of SLOCC equivalence classes.
  [F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A 65, 052112 (2002)]
- For many qubits, the practically meaningful classification is
  - (Fully) separable
  - Biseparable entangled
  - Genuine multipartite entangled

More than three qubits II

Definition

A state is (fully) separable if it can be written as $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}$.

Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

Here $|\Psi\rangle$ is an N-qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called genuine multi-partite entangled.

k-producibility/*k*-entanglement

Definition

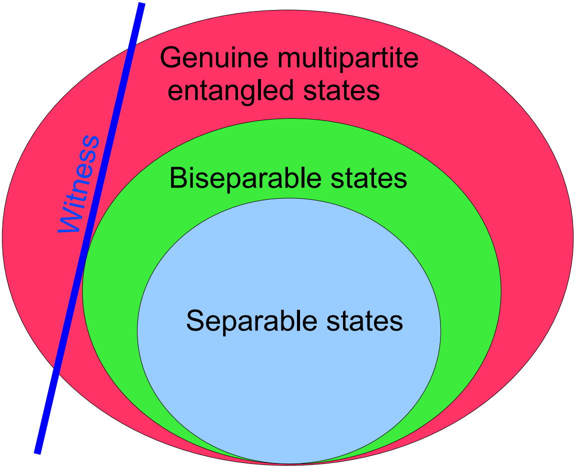
A pure state is *k*-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

where $|\Phi_l\rangle$ are states of at most k qubits. A mixed state is k-producible, if it is a mixture of k-producible pure states.

Convex sets for the multipartite case

- The idea of convex sets also work for the multi-qubit case: A state is biseparable if it can be obtained by mixing pure biseparable states.



Examples

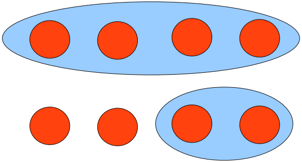
Examples

Two entangled states of four qubits:

$$|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

- The first state is genuine multipartite entangled, the second state is biseparable.

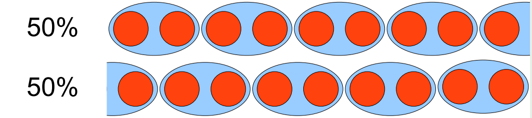


Other possible definition of genuine multipartite entanglement

- Alternative definition: a state is genuine multipartite entangled if it is inseparable with respect to all bipartitions.

Example

Mixture of the two biseparable states (chains of singlets)



It is inseparable with respect to all bipartitions.

- This state can be created in a two-qubit experiment.

Geometric measure of entanglement

Definition

For pure states, the geometric measure of entanglement is defined as

$$E_{\sin^2}(|\Psi\rangle) = 1 - \left(\max_{|\Psi_P\rangle \in \mathrm{PRODUCT}} \langle \Psi | \Psi_{\mathrm{P}} \rangle \right)^2 .$$

For mixed states, it is defined by a convex roof construction

$$E_{\sin^2}(\varrho) = \min_{\{|\Psi_k\rangle, p_k\}: \varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|} \sum_k p_k E_{\sin^2}(|\Psi_k\rangle).$$

- It is possible to calculate it for some pure states and for some mixed states.
  T.-C. Wei, P.M. Goldbart, Phys. Rev. A 68, 042307 (2003)

Bipartite measures

- Bipartite entanglement measures can also be used but they do not capture the complexity of multipartite entanglement.
- Examples:
  - negativity
  - entanglement of formation.

Outline

Physical systems

State-of-the-art in experiments

- 8 qubits with trapped cold ions (Nature, 2005)
- 10 qubits with photons (Nature Physics, 2010)

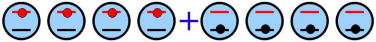
Main Challenges

- How to obtain useful information when only *local* measurements are possible?
- *In principle, the entanglement witness method has the advantage that only one observable, the entanglement witness, needs to be measured. In practice, the measurement of this observable may be done by a series of local measurements. ... At this point the advantage over basic state tomography becomes somewhat questionable.*
  (B. TERHAL, IBM Watson Research Center, 2002)

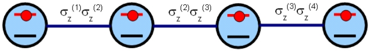
Interesting quantum states

Quantum states in experiments:

- Greenberger-Horn-Zeilinger (GHZ) state or "Schrödinger cat state”



- Cluster state, graph state (obtained in Ising spin chains)



- Symmetric Dicke states

- Singlet states
  $(\Delta J_l)^2 = 0 \qquad \text{for } j = x, y, z.$

Outline

Aims when designing a witness

Definition

An entanglement witness $\mathcal{W}$ is an operator that is positive on all separable (biseparable) states.

Thus, $\mathrm{Tr}(\mathcal{W}\varrho) < 0$ signals entanglement (genuine multipartite entanglement). Horodecki 1996; Terhal 2000; Lewenstein, Kraus, , Cirac, Horodecki 2002

There are two main goals when searching for entanglement witnesses:

Optimization → Large robustness to noise

Optimization → Few measurements

Robustness to noise

- A state mixed with white noise is given as

$$\varrho(p_{\text{noise}}) = (1 - p_{\text{noise}})\varrho + p_{\text{noise}}\varrho_{\text{noise}}$$

where p_{noise} is the ratio of noise and ϱ_{noise} is the noise. If we consider white noise then $\varrho_{\text{noise}} = \mathbb{1}/2^N$.

Definition

The noise tolerance of a witness $\mathcal{W}$ is characterized by the largest p_{noise} for which we still have

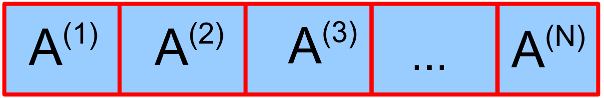
$$\text{Tr}(\mathcal{W}\varrho) < 0.$$

Only local measurements are possible

Definition

A single local measurement setting is the basic unit of experimental effort.

A local setting means measuring operator $A^{(k)}$ at qubit k for all qubits.



- All two-qubit, three-qubit correlations, etc. can be obtained.

$$\langle A^{(1)}A^{(2)}\rangle, \langle A^{(1)}A^{(3)}\rangle, \langle A^{(1)}A^{(2)}A^{(3)}\rangle ...$$

Decomposition of an operator

- All operators must be decomposed into the sum of locally measurable terms and these terms must be measured individually.
- For example,

$$
\begin{aligned}
|GHZ_3\rangle\langle GHZ_3| = & \frac{1}{8}(\mathbb{1} + \sigma_z^{(1)}\sigma_z^{(2)} + \sigma_z^{(1)}\sigma_z^{(3)} + \sigma_z^{(2)}\sigma_z^{(3)}) \\
& + \frac{1}{4}\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)} \\
& - \frac{1}{16}(\sigma_x^{(1)} + \sigma_y^{(1)})(\sigma_x^{(2)} + \sigma_y^{(2)})(\sigma_x^{(3)} + \sigma_y^{(3)}) \\
& - \frac{1}{16}(\sigma_x^{(1)} - \sigma_y^{(1)})(\sigma_x^{(2)} - \sigma_y^{(2)})(\sigma_x^{(3)} - \sigma_y^{(3)}).
\end{aligned}
$$

O. Gühne and P. Hyllus, Int. J. Theor. Phys. 42, 1001-1013 (2003). M. Bourennane et al., Phys. Rev. Lett. 92 087902 (2004)

- As N increases, the number of terms increases exponentially for projectors to quantum pure states.

Basic methods for designing witnesses

Three methods for designing witnesses:

- Projector witness, i.e., witness defined with the projector to a highly entangled quantum state
- Witness based on the projector witness
- Witness independent of the projector witness

Projector witness

- A witness detecting genuine multi-qubit entanglement in the vicinity of a pure state $|\Psi\rangle$ is

$$\mathcal{W}_{\Psi}^{(P)} := \lambda_{\Psi}^{2}\mathbb{1} - |\Psi\rangle\langle\Psi|,$$

  where λ is the maximum of the Schmidt coefficients for $|\Psi\rangle$, when all bipartitions are considered.
  M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 2004

- A symmetric witness operator can always be decomposed as

$$P = \sum c_k A_k \otimes A_k \otimes A_k \otimes ... \otimes A_k.$$

- For symmetric operators, the number of settings needed is increasing polynomially with the number of qubits.
  GT, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, New J. Phys. 2009

Projector witness II

- GHZ states (robustness to noise is $\frac{1}{2}$ for large N!)

$$\mathcal{W}^{(P)}_{\mathrm{GHZ}} := \frac{1}{2}\mathbb{1} - |GHZ_N\rangle\langle GHZ_N|.$$

- Cluster states

$$\mathcal{W}^{(P)}_{\mathrm{CL}} := \frac{1}{2}\mathbb{1} - |CL_N\rangle\langle CL_N|.$$

- Dicke state

$$\mathcal{W}^{(P)}_{\mathrm{D(N,N/2)}} := \frac{1}{2}\frac{N}{N-1}\mathbb{1} - |D_N^{(N/2)}\rangle\langle D_N^{(N/2)}|.$$

- W-state

$$\mathcal{W}^{(P)}_{\mathrm{W}} := \frac{N-1}{N}\mathbb{1} - |D_N^{(1)}\rangle\langle D_N^{(1)}|.$$

Witnesses based on the projector witness

- We construct witnesses that are easier to measure than the projector witness.

- Idea: If $\mathcal{W}^{(P)}$ is the projector witness and

$$\mathcal{W} - \alpha\mathcal{W}^{(P)} \geq 0$$

is fulfilled for some $\alpha > 0$, then $\mathcal{W}$ is also a witness.
GT and O. Gühne, Phys. Rev. Lett. and Phys. Rev. A 2005

Witnesses based on the projector witness II

Example

Witness requiring only two measurement settings for GHZ states

$$\mathcal{W}_{\text{GHZ}}^{(P)} := \frac{1}{2}\mathbb{1} - |GHZ_N\rangle\langle GHZ_N|$$

$$\leq \mathcal{W}_{\text{GHZ}}^{(P2)} := \mathbb{1} - \frac{1}{2}X_1X_2X_3...X_N - \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & ... & & \\ & & & 0 & \\ & & & & 1 \end{bmatrix}.$$

Measurement settings ⇒ [X X X X ...] [Z Z Z Z ...]

- Any state detected by $\mathcal{W}_{\text{GHZ}}^{(P2)}$ is also detected by $\mathcal{W}_{\text{GHZ}}^{(P)}$.
  GT and O. Gühne, Phys. Rev. Lett. and Phys. Rev. A 2005

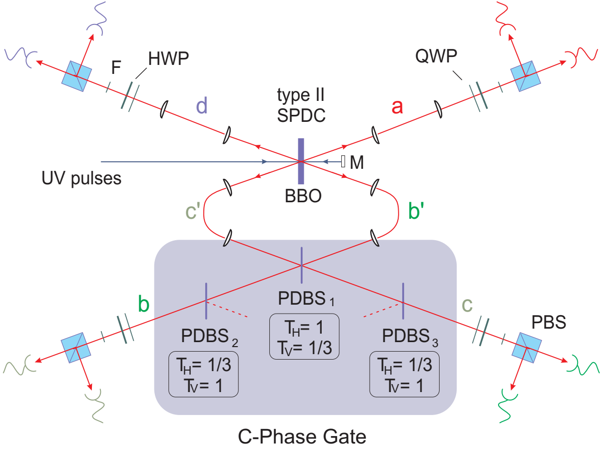
Witnesses independent from the projector witness

- Witnesses without any relation to the projector witness.
- With an easily measurable operator M, we make a witness of the form
$$\mathcal{W} := c\mathbb{1} - M,$$
where c is some constant.
- We have to set c to
$$c = \max_{|\Psi\rangle \in \mathcal{B}} \langle M \rangle_{|\Psi\rangle},$$
where $\mathcal{B}$ is the set of biseparable states. This problem is typically hard to solve.

Outline

An experiment: Cluster state with photons

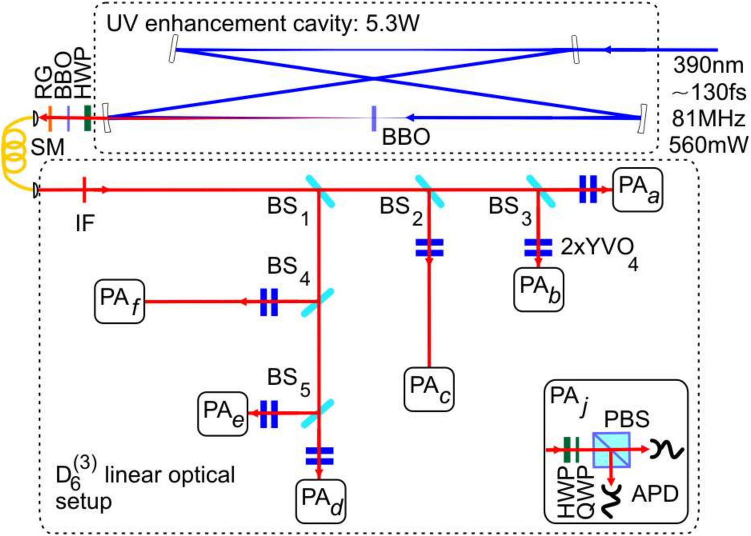
Experiment for creating a four-photon cluster state (Weinfurter group, 2005)



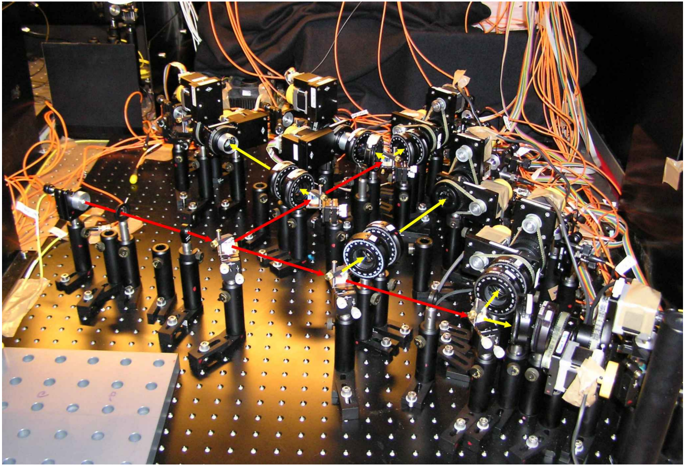
An experiment: Cluster state with photons II

- Note: the experiment works with conditional detection.
- So far the largest experiment is with 6 photons, and with 10 qubits.
- 1 photon can encode more than 1 qubit: hyperentanglement.

An experiment: Dicke state with photons

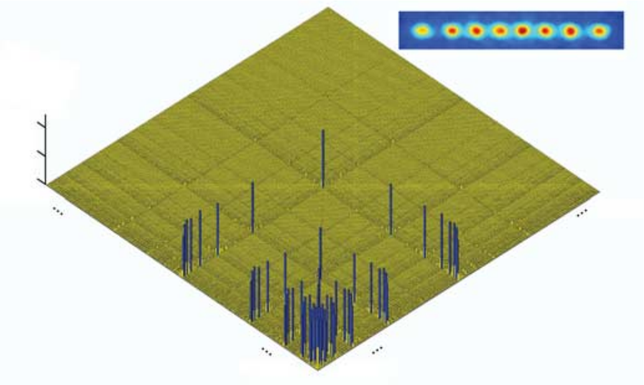


A photo of a real experiment (six-photon Dicke state, Weinfurter group, 2009):



Experiment: W-state with ions

- Experimental observation of an 8-qubit W-state with trapped ions.



H. Haeffner, W. Haensel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Koerber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Gühne, W. Dür, R. Blatt, Nature 438, 643-646 (2005).

Quantum state tomography

- The density matrix can be reconstructed from 3^N measurement settings.

- The measurements are

1. XXXX

2. XXXY

3. XXXZ

...

3^4. ZZZZ

- Note again that the number of measurements scales exponentially in N.

Outline

Physical systems

State-of-the-art in experiments

- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with 10^6 - 10^{12} atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.

Many-particle systems

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2$$

variances.

Outline

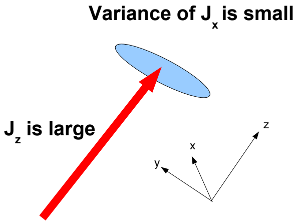
Spin squeezing

Definition

Uncertainty relation for the spin coordinates

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2.$$

If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_z\rangle|$ then the state is called spin squeezed (mean spin in the z direction!).
[M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993)]



Spin squeezing II

Definition

Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

- Application: Quantum metrology, magnetometry.

[A. Sørensen *et al.*, Nature **409**, 63 (2001)]

Complete set of the generalized spin squeezing criteria

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle),$$

$$\vec{K} := (\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle).$$

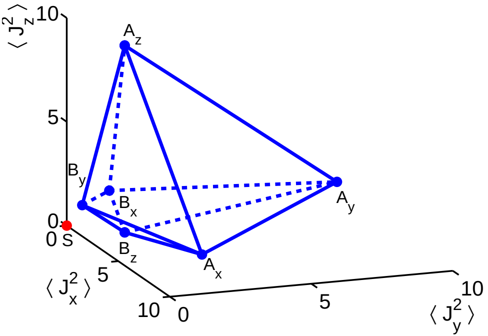
- Then any state violating the following inequalities is entangled

$$\begin{aligned} \langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle &\leq N(N+2)/4, \\ (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq N/2, \\ \langle J_k^2\rangle + \langle J_l^2\rangle - N/2 &\leq (N-1)(\Delta J_m)^2, \\ (N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2\rangle + N(N-2)/4. \end{aligned}$$

where k, l, m takes all the possible permutations of x, y, z.
[GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007]

The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z}\rangle$, describe a polytope in the $\langle J^2_{x/y/z}\rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!



The derivation of such criteria

- The derivation of such criteria is partly based on entanglement detection with uncertainty relations.
- For a multi-qubit pure product state $|\Psi_{\rm P}\rangle = \bigotimes_k |\psi_k\rangle$we have

$$(\Delta J_l)^2 = \sum_k (\Delta j_l^{(k)})^2{}_{\psi_k}.$$

- Hence,

$$\sum_{l=x,y,z} (\Delta J_l)^2_{|\Psi_{\rm P}\rangle} = \sum_{l=x,y,z} \sum_{k=1}^{N} (\Delta J_l)^2_{|\Psi_k\rangle} =$$

$$\frac{1}{4} \sum_{k=1}^{N} (3 - \langle \sigma_x^{(k)} \rangle^2 - \langle \sigma_y^{(k)} \rangle^2 - \langle \sigma_z^{(k)} \rangle^2) = \frac{N}{2}.$$

- Due to the concavity of the variance, for mixed separable states we have

$$\sum_{l=x,y,z} (\Delta J_l)^2 \geq \frac{N}{2}.$$

[G. Tóth, PRA 2004; O. Gühne, PRL 2004; H.F. Hofmann and S. Takeuchi, PRA 2003.]

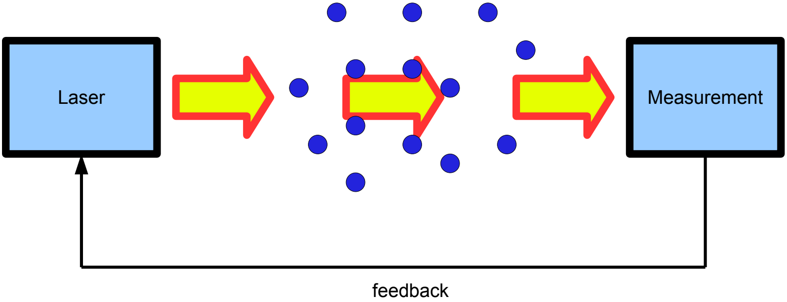
Outline

The physical system

- Bose Eisnetin condensate of atoms: the atoms interact with each other
- Cold gases: the atoms do not interact with each other

The physical system II

- Cold gases: Rb atoms + light

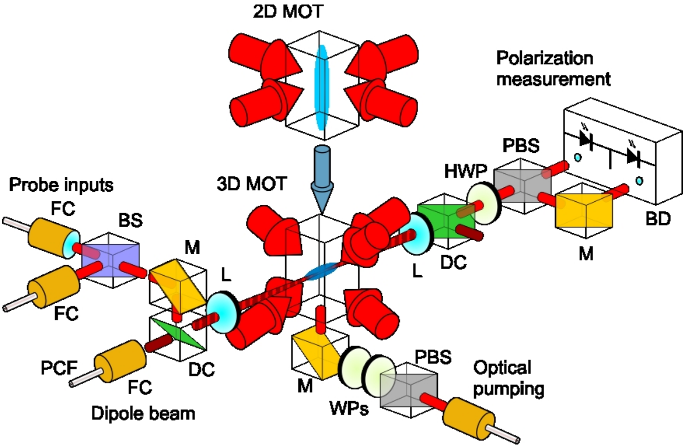


Experimental details

- Atoms interact with light. The light is measured, projecting the atoms into a squeezed state.
- Room temperature experiments: 10^{12} atoms
  [B Julsgaard, A Kozhekin, ES Polzik, Nature 2001].
  - Vapor cells
- Cold atom experiments: 10^{6} atoms.
  - Laser cooling, sample in an optical dipole trap.
  - Atoms are transferred from a MOT to a dipole trap.

An experiment

Spin squeezing in a cold atomic ensemble (not BEC!)



Picture from M.W. Mitchell, ICFO, Barcelona.

Outline

Quantum Fisher information

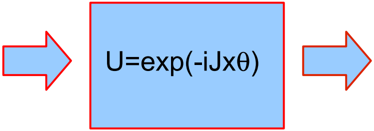
- One of the important applications of entangled multipartite quantum states is sub-shotnoise metrology.
  [V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).]

- Multipartite entanglement, not simple nonseparability, is needed for extreme spin squeezing, which can be applied in spectroscopy and atomic clocks.
  [A.S. Sørensen and K. Mølmer, Phys. Rev. Lett. 86, 4431 (2001).]

- Not all entangled states are useful for phase estimation, at least in a linear interferometer.
  [P. Hyllus, O. Gühne, and A. Smerzi, arXiv:0912.4349.]

Quantum Fisher information II

- Let us consider the following process:



- The dynamics described above is $\varrho_{\text{out}} = e^{-i\theta J_{\vec{n}}} \varrho e^{+i\theta J_{\vec{n}}}$.
- We would like to determine the angle θ by measuring ϱ_{out}.

Quantum Fisher information III

Quantum Cramér-Rao bound

For such a linear interferometer the phase estimation sensitivity is limited by the Quantum Cramér-Rao bound as

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q[\varrho, J_{\vec{n}}]}},$$

where F_Q is the quantum Fisher information, ϱ is a quantum state and $J_{\vec{n}}$ is a component of the collective angular momentum in the direction $\vec{n}$.

[C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976);
A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982).]

Quantum Fisher information IV

- The quantum Fisher information is the supremum of the following [Braunstein, Caves, 1994]

$$F(\varrho(\theta), \{E(\xi)\}) = \int \frac{[\mathrm{Tr}\varrho(\theta)' E(\xi)]^2}{\mathrm{Tr}\varrho(\theta) E(\xi)} d\xi.$$

- In another context, there are several possible Fisher informations. The Braunstein-Caves's one is the *minimal* Fisher information.

$$F[\varrho, X] = \sum_{ij} \frac{2(\lambda_i - \lambda)^2}{\lambda_i + \lambda_j} |X_{ij}|^2.$$

[D. Petz, Monotone metrics on matrix spaces, Linear Algebra Appl. 244(1996), 81–96;
D. Petz and Cs. Sudár, World Scientific, 1999;
D. Petz and C. Ghinea, arXiv:1008.2417.]

Outline

Properties of the Quantum Fisher information

For calculating many quantities, it is sufficient to know that following two relations.

1. For a pure state ϱ, we have $F[\varrho, J_l] = 4(\Delta J_l)^2_\varrho$.
2. $F[\varrho, J_l]$ is convex in the state, that is $F[p_1\varrho_1 + p_2\varrho_2, J_l] \leq p_1 F[\varrho_1, J_l] + p_2 F[\varrho_2, J_l]$.

From these two statements, it also follows that $F[\varrho, J_l] \leq 4(\Delta J_l)^2_\varrho$.

[C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976);
A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982);
S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. 72, 3439 (1994); L. Pezzé and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).]

Properties of the Quantum Fisher information II

For computing the Fisher information numerically, we recall that the quantum Fisher information $F_Q[\varrho, J_{\vec{n}}]$ for any $\vec{n}$ can be given as

$$F_Q[\varrho, J_{\vec{n}}] = 4\vec{n}^T \Gamma_C \vec{n}.$$

Here, the Γ_C matrix is defined as

$$[\Gamma_C]_{ij} = \frac{1}{2} \sum_{l,m} (\lambda_l + \lambda_m) \left(\frac{\lambda_l - \lambda_m}{\lambda_l + \lambda_m} \right)^2 \langle l|J_i|m\rangle\langle m|J_j|l\rangle,$$

where the sum is over the terms for which $\lambda_l + \lambda_m \neq 0$, and the density matrix has the decomposition

$$\varrho = \sum_k \lambda_k |k\rangle\langle k|.$$

For pure states, and $[\Gamma_C]_{ij} = \langle J_i J_j + J_j J_i\rangle/2 - \langle J_i J_j\rangle$.

[P. Hyllus, O. Gühne, and A. Smerzi, arXiv:0912.4349.]

Outline

Quantum Fisher information and entanglementand

Pezzé, Smerzi, PRL 2009

For N-qubit separable states, the values of $F_Q[\varrho, J_l]$ for $l = x, y, z$ are bounded as

$$F_Q[\varrho, J_l] \leq N.$$

Here, $J_l = \frac{1}{2}\sum_{k=1}^{N} \sigma_l^{(k)}$ where $\sigma_l^{(k)}$ are the Pauli spin matrices for qubit (k). The maximum for the left-hand side is N^2.

Thus, for separable states

$$\Delta\theta \geq \frac{1}{\sqrt{N}},$$

while for entangled states

$$\Delta\theta \geq \frac{1}{N}.$$

Quantum Fisher information and entanglement II

Observation 1

For N-qubit separable states, the values of $F_Q[\varrho, J_l]$ for $l = x, y, z$ are bounded as

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq 2N. \qquad (2)$$

- Later we will also show that Eq. (2) is a condition for the average sensitivity of the interferometer. All states violating Eq. (2) are entangled.

[GT, arxiv:1006.4368; P. Hyllus et al., arXiv:1006.4366.]

Quantum Fisher information and entanglement III

Observation 2

For quantum states, the Fisher information is bounded by above as

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq N(N+2). \tag{3}$$

Greenberger-Horne-Zeilinger (GHZ) states and N-qubit symmetric Dicke states with $\frac{N}{2}$ excitations saturate Eq. (3).

- The above symmetric Dicke state has been investigated recently due to its interesting entanglement properties. It has also been noted that above Dicke state gives an almost maximal phase measurement sensitivity in two orthogonal directions.
- In general, pure symmetric states for which $\langle J_l \rangle = 0$ for $l = x, y, z$ saturate Eq. (3).

[GT, arxiv:1006.4368; P. Hyllus et al., arXiv:1006.4366.]

Quantum Fisher information and multipartite entanglement

Next, we will consider k-producible or k-entangled states:

Observation 3

For N-qubit k-producible states states, the sum of three Fisher information terms is bounded from above by

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq nk(k+2) + (N-nk)(N-nk+2).$$

where n is the integer part of $\frac{N}{k}$. For the $k = N-1$ case, this bound can be improved

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq N^2 + 1. \tag{4}$$

Eq. (4) is also the inequality for biseparable states. Any state that violates Eq. (4) is
genuine multipartite entangled.

Quantum Fisher information and multipartite entanglement II

Fact

Genuine multipartite entanglement, not simple nonseparability is needed to achieve maximum sensitivity in a linear interferometer.

Quantum Fisher information and multipartite entanglement III

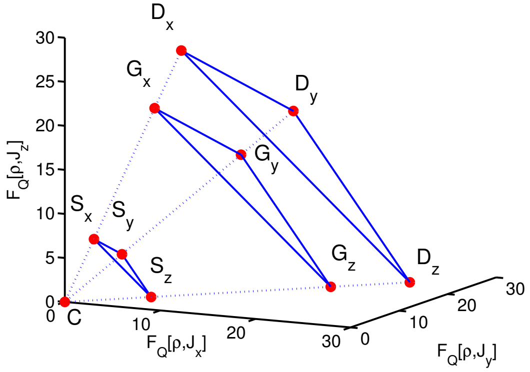


Figure: Interesting points in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space for $N = 6$ particles. Points corresponding to separable states satisfy Eq. (2) and are not above the $S_x - S_y - S_z$ plane. Biseparable states satisfy Eq. (4) and are not above the $G_x - G_y - G_z$ plane.

Proof of Observation 1

First, we shown that Observation 1 is true for pure states. For every N-qubit pure product state of the form

$$|\Psi_{\rm P}\rangle = \otimes_{k=1}^{N} |\Psi_k\rangle$$

we have

$$\sum_{l=x,y,z} (\Delta J_l)^2_{|\Psi_{\rm P}\rangle} = \sum_{l=x,y,z} \sum_{k=1}^{N} (\Delta J_l)^2_{|\Psi_k\rangle} =$$

$$\frac{1}{4} \sum_{k=1}^{N} (3 - \langle \sigma_x^{(k)} \rangle^2 - \langle \sigma_y^{(k)} \rangle^2 - \langle \sigma_z^{(k)} \rangle^2) = \frac{N}{2}.$$

For mixed states, $\sum_{l=x,y,z} F_Q[\varrho, J_l] \le 2N$ follows from the convexity of the Fisher information. This finishes the proof.

[G. Tóth, Phys. Rev. A 69, 052327 (2004).]

Which part of the space corresponds to quantum states

- We discuss which part of the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space contains points corresponding to states with different degree of entanglement.

- This is important, since apart from finding inequalities for states of various types of entanglement, we have to show that there are states that fulfill these inequalities.

Which part of the space corresponds to quantum states

Let us see first the interesting points of the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space and the corresponding quantum states:

- A completely mixed state

$$\varrho_C = \frac{\mathbb{1}}{2^N}.$$

  corresponds to the point $C(0,0,0)$.

- States corresponding to the points $S_x(0,N,N), S_y(N,0,N), S_z(0,N,N)$ are

$$|\Psi\rangle_{S_l} = |+\frac{1}{2}\rangle_l^{\otimes N/2} \otimes |-\frac{1}{2}\rangle_l^{\otimes N/2}$$

  for $l = x, y, z$.

Which part of the space corresponds to quantum states II

- For the point $D_z(N(N+2)/2, N(N+2)/2, 0)$, a corresponding quantum state is an N-qubit symmetric Dicke state with $N/2$ excitations in the z basis.

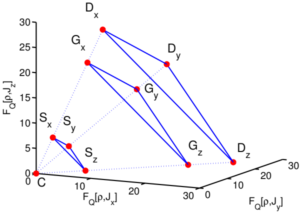
$$|\mathcal{D}_N^{(N/2)}\rangle = \binom{N}{N/2}^{-\frac{1}{2}} \sum_k \mathcal{P}_k\{|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}}\},$$

  where $\sum_k \mathcal{P}_k$ denotes summation over all possible permutations.

- For the point (N, N, N^2), a corresponding quantum state is an N-qubit GHZ states in the z basis

$$|\Psi\rangle_{GHZ_z} = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right).$$

Which part of the space corresponds to quantum states III



- For all points in the S_x, S_y, S_z polytope, there is a corresponding pure product state for even N.

- For given $F[\varrho, J_l]$ for $l = x, y, z$, such a state is defined as

$$\varrho = \left[\frac{\mathbb{1}}{2} + \frac{1}{2}\sum_{l=x,y,z} c_l \sigma_l\right]^{\otimes N/2} \otimes \left[\frac{\mathbb{1}}{2} - \frac{1}{2}\sum_{l=x,y,z} c_l \sigma_l\right]^{\otimes N/2},$$

where $c_l^2 = 1 - \frac{F_Q[\varrho, J_l]}{N}$, where $\sum_l c_l^2 = 1$.

Which part of the space corresponds to quantum states V

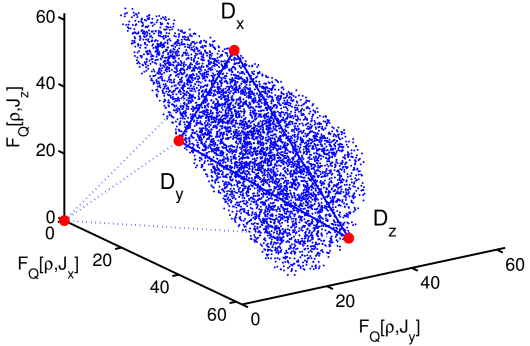


Figure: Randomly chosen points in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space corresponding to states of the form
$|\Psi(\alpha_x, \alpha_y, \alpha_z)\rangle = \alpha_x|\mathcal{D}_N^{(N/2)}\rangle_x + \alpha_y|\mathcal{D}_N^{(N/2)}\rangle_y + \alpha_z|\mathcal{D}_N^{(N/2)}\rangle_z$, for $N = 8$. All the points are in the plane of D_x, D_y and D_z.

Which part of the space corresponds to quantum states VI

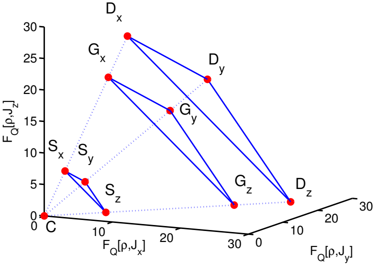
- Three-dimensional polytopes. The points corresponding to the mixed state are on a curve in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])\text{-space}$. In the general case, this curve is not a straight line. For the case of mixing a pure state with the completely mixed state the curve is a straight line. Such a state is defined as

$$\varrho^{(\text{mixed})}(p) = p\varrho + (1-p)\frac{\mathbb{1}}{2^N}$$

- Using the formula for Γ_C, after simple calculations we have

$$\Gamma_C^{(\text{mixed})}(p) = \frac{p^2}{p + (1-p)2^{-(N-1)}}\Gamma_C^{(\varrho)}.$$

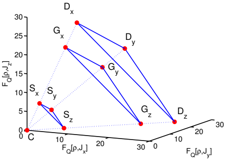
Which part of the space corresponds to quantum states VII



Observation 5. If N is even, then there is a separable state for each point in the S_x, S_y, S_z, C polytope.

Proof. This is because there is a pure product state corresponding to any point in the S_x, S_y, S_z polytope. When mixing any of these states with the completely mixed state, we obtain states that correspond to points on the line connecting the pure state to point C.

Which part of the space corresponds to quantum states VIII



Observation 6. If N is divisible by 4, then for all the points of the $D_x, D_y, D_z, G_x, G_y, G_z$ polytope, there is a quantum state with genuine multipartite entanglement.

Proof. There is a quantum state for all points in the D_x, D_y, D_z polytope. Mixing them with the completely mixed state, states corresponding to all points of the C, D_x, D_y, D_z polytope can be obtained. Based on Observation 2, states corresponding to the points in the $D_x, D_y, D_z, G_x, G_y, G_z$ polytope are genuine multipartite entangled.

Finally, note that all the quantum states we presented in this section have a diagonal Γ_C matrix.

Summary

- We discussed entanglement detection in multipartite systems.
- We considered
  - systems with few particles in which the particles could be individually addressed.
  - systems with very many particles, without the possibility of individual addressing

Review: O. Gühne and GT, “Entanglement detection”,

Physics Reports 474, 1-75 (2009).

THANK YOU FOR YOUR ATTENTION!