Witnessing *Genuine* Many-qubit Entanglement with Very Few Local Measurements

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Outline

- Genuine multi-qubit entanglement
- Entanglement detection with entanglement witnesses
- Witness based on projectors
- Our proposal: witness with few local measurements (for GHZ & cluster states)
- Connection to Bell inequalities

Genuine multi-qubit entanglement

- Genuine three-qubit entanglement

$$|000\rangle + |111\rangle$$

- Biseparable entanglement

$$|001\rangle + |111\rangle = \left(|00\rangle + |11\rangle\right)|1\rangle$$

- A mixed entangled state is *biseparable* if it is the mixture of biseparabe states (of possibly different partitions).

Entanglement witnesses I

- Bell inequalities

  Classical: no knowledge of quantum mechanics is used to construct them.

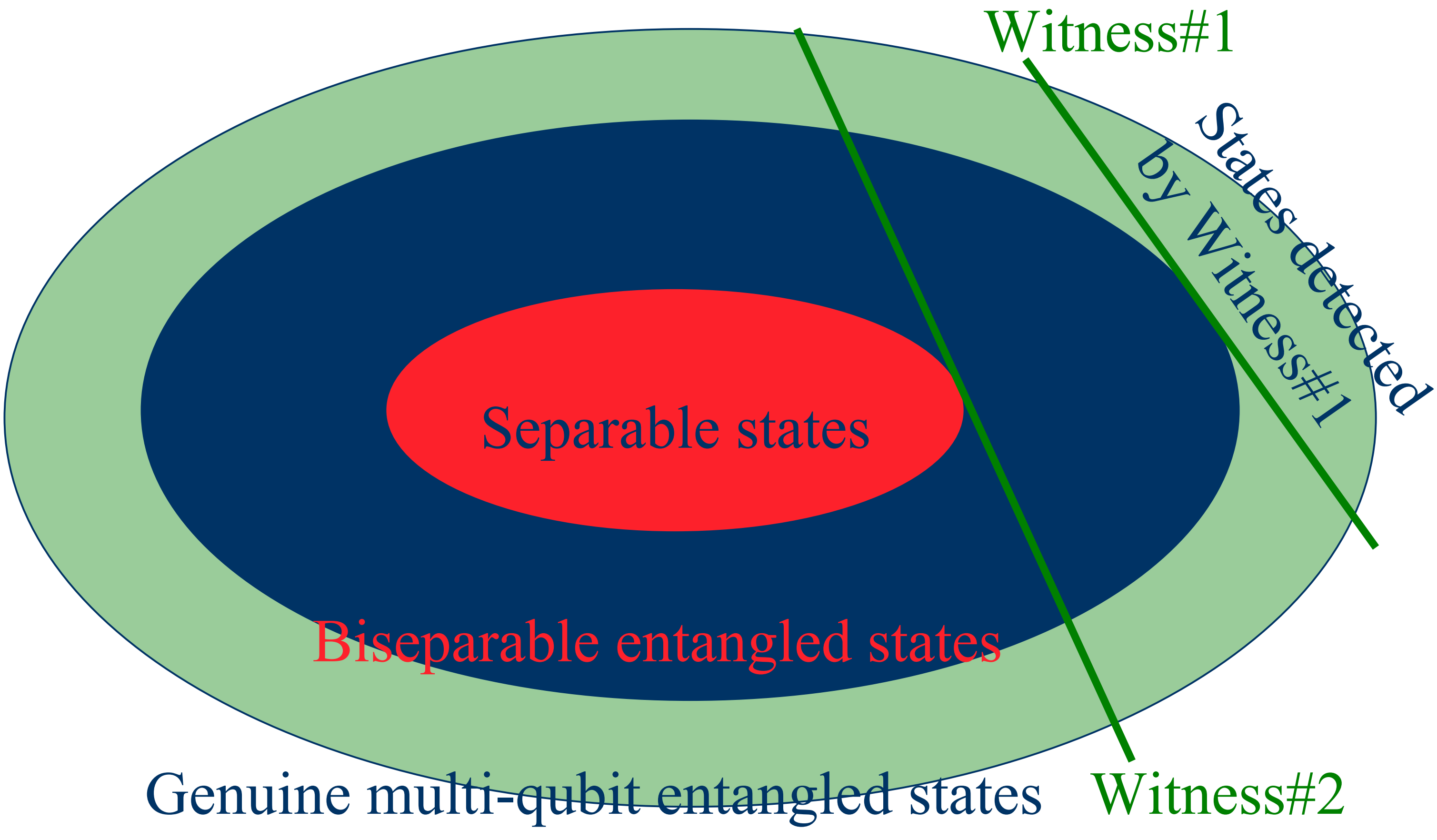
- Entanglement witnesses

  Knowledge of QM is used for constructing them.

Entanglement witnesses II

- Entanglement witnesses are observables which have
  - positive expectation values for separable states
  - negative expectation values for *some* entangled states.
- Witnesses can be constructed which detect entangled states close to a state chosen by us.
- Witnesses can be constructed which detect *only genuine multi-party* entanglement.

Entanglement witnesses III



Entanglement witnesses IV

- It is possible to construct witnesses for detecting entangled states close to a particular state with a projector. E.g.,

$$W_{GHZN}^{PROJ} = \frac{1}{2} \bullet 1 - |GHZ_N\rangle\langle GHZ_N|$$

detects N-qubit entangled states close to an N-qubit GHZ state.

Entanglement witnesses V

- So if

$$\left\langle W_{GHZN}^{PROJ} \right\rangle < 0$$

then the system is genuinely multi-qubit entangled.

- Question: how can we measure the witness operator?

Decomposing the witness

- For an experiment, the witness must be decomposed into locally measurable terms

$$W_{GHZ3}^{PROJ} = \frac{1}{8}(3\cdot 1 - \sigma_z^1\sigma_z^2 - \sigma_z^1\sigma_z^3 - \sigma_z^2\sigma_z^3 - 2\sigma_x^1\sigma_x^2\sigma_x^3)$$

$$+\frac{1}{4}\left(\sigma_x^1 + \sigma_y^1\right)\left(\sigma_x^2 + \sigma_y^2\right)\left(\sigma_x^3 + \sigma_y^3\right)$$

$$+\frac{1}{4}\left(\sigma_x^1 - \sigma_y^1\right)\left(\sigma_x^2 - \sigma_y^2\right)\left(\sigma_x^3 - \sigma_y^3\right)$$

- See O. Gühne, P. Hyllus, quant-ph/0301162; M. Bourennane et. al., PRL 92 087902 (2004).

Main topic of the talk: How can one decrease the number of local terms

- As the number of qubits increases, the number of local terms increases exponentially. Similar thing happens to Bell inequalities for the GHZ state.
- Q: How can we construct entanglement witnesses with few locally measurable terms?

Entanglement witnesses based on the stabilizer formalism

Stabilizer witnesses

- We propose new type of witnesses. E.g., for three-qubit GHZ states

$$W_{GHZ3} = 2\bullet 1 - \sigma_x^1\sigma_x^2\sigma_x^3 - \sigma_z^1\sigma_z^2 - \sigma_z^2\sigma_z^3$$

- All the three terms are +1 for the GHZ state.

Stabilizer witnesses II

- General method for constructing witnesses for states close to $|\Psi\rangle$

$$W = c \cdot 1 - \sum_{k=1}^{N} S_k$$

- Here S_k stabilize $|\Psi\rangle$

$$|\Psi\rangle = S_k |\Psi\rangle$$

Stabilizing operators $|\Psi\rangle = S_k |\Psi\rangle$

- For an N-qubit GHZ state

$$S_1 = \sigma_x^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} \ldots \sigma_x^{(N)},$$

$$S_k = \sigma_z^{(k-1)} \sigma_z^{(k)}; \quad k = 2, 3, \ldots, N.$$

For an N-qubit cluster state

$$S_1 = \sigma_x^{(1)} \sigma_z^{(2)},$$

$$S_k = \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)}; \quad k = 2, 3, \ldots, N-1,$$

$$S_N = \sigma_z^{(N-1)} \sigma_x^{(N)}.$$

Cluster state

- Obtained from Ising spin chain dynamics
- For N=3 qubits it is equivalent to a GHZ state
- For N=4 qubits it is equivalent to

$$|C4\rangle = |0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle$$

- See Briegel, Raussendorf, PRL 86, 910 (2001).

Stabilizer witnesses III

- Characteristics for our N-qubit entanglement witnesses
  - N locally measurable terms
  - Usually 2 (!!) measurement settings
  - Tolerates noise $p_{noise}<1/N$
  - Noise tolerance can be improved if more than N terms are included

Stabilizer witnesses IV

- Witness for N-qubit GHZ state

$$W_{GHZN} = (N-1) - \prod_{k=1}^{N} \sigma_x^{(k)} - \sum_{k=1}^{N-1} \sigma_z^{(k)} \sigma_z^{(k+1)}$$

- Witness for N-qubit cluster state

$$W_{clN} = (N-1) - \sigma_x^{(1)} \sigma_z^{(2)} - \sigma_z^{(k-1)} \sigma_x^{(k)} - \sum_{k=2}^{N-1} \sigma_z^{(k-1)} \sigma_x^{(k)} \sigma_z^{(k+1)}$$

Stabilizer witnesses V

- Why do these witnesses detect genuine N-qubit entanglement? Because

$$W_{GHZN} - 2W_{GHZN}^{PROJ} \geq 0 \qquad \left(W_{GHZN}^{PROJ} = \frac{1}{2}\bullet 1 - |GHZ_N\rangle\langle GHZ_N| \right)$$

- Any state detected by our witness is also detected by the projector witness. Later detects genuine N-qubit entanglement.

Noise

- In an experiment the GHZ state is never prepared perfectly

$$\rho = (1 - p_{noise}) |GHZ_3\rangle\langle GHZ_3| + p_{noise}\rho_{totally\_mixed}$$

- For each witness there is a noise limit. For a noise larger than this limit the GHZ state is not detected as entangled.

Improving noise tolerance

- Noise tolerance: 33% (3 terms)

$$W_{GHZ3} = 2 - \sigma_x^1 \sigma_x^2 \sigma_x^3 - \sigma_z^1 \sigma_z^2 - \sigma_z^2 \sigma_z^3$$

- Noise tolerance: 50% (4 terms) Bell inequality!

$$W'_{GHZ3} = 2 - \sigma_x^1 \sigma_x^2 \sigma_x^3 + \sigma_y^1 \sigma_y^2 \sigma_x^3 + \sigma_x^1 \sigma_y^2 \sigma_y^3 + \sigma_y^1 \sigma_x^2 \sigma_y^3$$

- Noise tolerance: 57% (7 terms) Projector!!

$$W_{GHZ3}^{PROJ} = W'_{GHZ3} + 1 - \sigma_z^1 \sigma_z^2 - \sigma_z^2 \sigma_z^3 - \sigma_z^1 \sigma_z^3$$

Some interesting connections to Bell inequalities

Mermin's inequality

- Mermin's inequality

$$W'_{GHZ3} = 2 - \sigma_x^1 \sigma_x^2 \sigma_x^3 + \sigma_y^1 \sigma_y^2 \sigma_x^3 + \sigma_x^1 \sigma_y^2 \sigma_y^3 + \sigma_y^1 \sigma_x^2 \sigma_y^3$$

- It does detect genuine three-qubit entanglement in contrast to Seevinck, Uffink, PRA 65, 012107 (2001).

Mermin's inequality II

- Seevink & Uffink say that three-qubit entanglement is sure only if

$$\left\langle \sigma_x^1\sigma_x^2\sigma_x^3 - \sigma_y^1\sigma_y^2\sigma_x^3 - \sigma_x^1\sigma_y^2\sigma_y^3 - \sigma_y^1\sigma_x^2\sigma_y^3 \right\rangle > 2\sqrt{2}$$

- This is not correct. The bound is 2. The problem is related to Bell inequlities.

Mermin's inequality III

- Choosing arbitrary two observables at each site, one has for biseparable states

$$E(a'b'c)+E(ab'c')+E(a'bc')-E(abc)\leq 2\sqrt{2}$$

- Gisin, Bechmann-Pasquinucci, Phys. Lett. A 246, 1 (1998): Examples of states saturating the inequality are shown *if c*'=*c*!

Mermin's inequality IV

- If, however, a and a' correspond to orthogonal directions

$$E(a'b'c)+E(ab'c')+E(a'bc')-E(abc)\leq 2$$

- Simple proof for biseparable states:

$$\left\langle \sigma_x^1\sigma_x^2\sigma_x^3-\sigma_y^1\sigma_y^2\sigma_x^3-\sigma_x^1\sigma_y^2\sigma_y^3-\sigma_y^1\sigma_x^2\sigma_y^3 \right\rangle_\Psi$$

$$\leq 4\left\langle \left|GHZ_N\right\rangle\left\langle GHZ_N\right| \right\rangle_\Psi = 4\left|\left\langle GHZ_N \middle| \Psi\right\rangle\right|^2 \leq 2$$

Ardehali's inequality for 0110+1001

- Condition for 4qubit entanglement (16 terms)

$$\frac{1}{2}\left(xxx - xyy + yxy + yyx\right)(a+b) +$$

$$\frac{1}{2}(yyy + xyx + xxy - yxy)(a-b) > 4$$

$$a = \frac{x+y}{\sqrt{2}} \qquad b = \frac{x-y}{\sqrt{2}}$$

Ardehali's inequality II

- Entanglement witness from A's inequality

$$W_A := 4 \bullet 1 - \frac{1}{\sqrt{2}}(xxxx - xyyx + yxyx + yyxx$$

$$+ yyyy + xyxy + xxyy - yxxy)$$

- 8 terms

This term is 0 for the GHZ state.
The other terms are +1/-1.

Ardehali's inequality III

- Can the bound be smaller for 4-qubit entanglemet? YES

$$\left(W_A - 0.7 \bullet 1\right) - const \times W_{GHZ4}^{PROJ} \geq 0$$

- Thus the bound can be lowered from 4 to ~3.3.

Ardehali's inequality IV

- Can we use less terms? YES

$$W_A' := 4 \cdot 1 - xxxx + xyyx - yyxx$$
$$- xyxy - xxyy + yxxy$$

- This witness tolerates 33% noise and would detect the state in Zhao et. al., PRL 2003 as entangled. (All terms are around 0.7.)

W state ($|W\rangle = |100\rangle + |010\rangle + |001\rangle$)

- The W state does not fit the stabilizer framework.Thus there are no locally measurable S_k's such that $|W\rangle = S_k |W\rangle$
- But the W state is uniquely defined by

$$\frac{1}{4}\left(\sigma_x^1\sigma_x^2 + \sigma_x^1\sigma_x^3 + \sigma_x^2\sigma_x^3 + \sigma_y^1\sigma_y^2 + \sigma_y^1\sigma_y^3 + \sigma_y^2\sigma_y^3\right)|W\rangle$$

$$= |W\rangle$$

$$\sigma_z^1\sigma_z^2\sigma_z^3 |W\rangle = |W\rangle$$

Wstate II

- Ad-hoc witness (6 terms, 20% noise)

$$\mathcal{W}_{W3} = (1+\sqrt{5}) - \sum_{k\neq l} \sigma_x^k \sigma_x^l - \sum_{k\neq l} \sigma_y^k \sigma_y^l$$

- Detects entangled states around

$$\left|W\right\rangle = \left|100\right\rangle + \left|010\right\rangle + \left|001\right\rangle$$

$$\left|\overline{W}\right\rangle = \left|011\right\rangle + \left|101\right\rangle + \left|110\right\rangle$$

Summary

- Detection of genuine N-qubit entanglement was considered with few local measurements.
- The methods detect entangled states close to N-qubit GHZ and cluster states.
- Home page:
  http://www.mpq.mpg.de/Theorygroup/CIRAC/people/toth

- *************** THANK YOU!!! *************