Two Measurement Settings can Suffice to Verify Multipartite Entanglement

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Abstract. We present entanglement witnesses for detecting *genuine* multi-qubit entanglement. Our constructions are robust against noise and require only two *local* measurement settings, independent of the number of qubits. Thus they allow us to verify entanglement of many qubits in experiments while requiring only a small effort. In contrast, usual methods need an effort which increases exponentially with the number of qubits. The witnesses detect states close to GHZ states and cluster states.

INTRODUCTION

Entanglement, a strange phenomenon of the quantum world, has been known since the first half of the previous century. Recently, new insight on entanglement was gained through quantum information science which connects physics with algorithmic theory. In this context, besides asking "What are the characteristics of an entangled state?", we can also ask "What kind of tasks can be done with entangled states?" or "What types of entangled states can be created?"

Questions of the second type lead to the classification of multi-partite entangled states. For two-qubits it is enough to say: "This state is entangled" or "This state is separable". From an ensemble of two-qubit systems it is always possible to distill, with local operations and classical communication (LOCC), a maximally entangled singlet state, if the corresponding density matrix is entangled.

But already for three qubits, the situation is much more complicated. First of all, we have to differentiate the case when two qubits are entangled and the third is not entangled with them (e.g., $|\phi_1\rangle = |0\rangle(|00\rangle + |11\rangle)/\sqrt{2}$) from real three qubit entanglement (e.g., $|\phi_2\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$). Moreover, given two genuine tripartite entangled three qubit states, one may ask whether it is possible to convert a state into another one using only LOCC. Surprisingly, it turns out that not all pure states with *genuine* three-qubit entanglement are equivalent under LOCC, not even stochastically. In fact, there are two inequivalent classes, the W and the GHZ class [1]. This classification can be extended to mixed states [2] where the W class is inclusive of the GHZ class. For pure four qubit states, the number of equivalence classes is infinite [3] and the extension of the classification to mixed states does not seem to be useful.

Thus for pure states of many qubits we are left with three qualitatively different cases: The *fully separable* states are product states with no correlations between the parties. For the *biseparable* states there always exists one partition of the qubits into two parties, which are separable and not correlated. However, the qubits inside one party may be entangled. For *genuine multipartite entangled* states no such splitting can be found. A *mixed* state is biseparable (respectively, fully separable) if it can be constructed by mixing biseparable (respectively, fully separable) pure states.

In this paper we will describe a method how to detect genuine N-qubit entanglement around GHZ (Greenberger-Horne-Zeilinger, [4]) and cluster states [5]. Besides being theoretically interesting, the motivation for detecting multi-qubit entanglement also comes from the side of the experimentalists. Recently, several experiments succeeded in creating various multi-qubit states with photons [6], trapped ions [7] or cold atoms trapped with optical lattices [8]. For all these experiments it is crucial to prove that the quantum state is genuine multipartite entangled: A multi-qubit experiment presents something qualitatively new only if provably more than two qubits are entangled.

Detecting genuine multi-qubit entanglement is a difficult problem since it is inherently *nonlocal*, while in most

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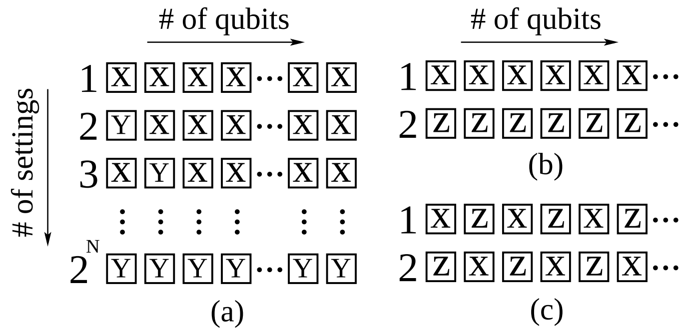

FIGURE 1. (a) The 2^N measurement settings needed using Bell inequalities for detecting multi-qubit entanglement close to a GHZ state. For each qubit the measured spin component is indicated. (b) The two settings needed for our witnesses detecting genuine multi-qubit entanglement close to GHZ and (c) cluster states.

experiments only *local* measurements are possible. One option is using Bell inequalities. These indicate the violation of local realism, a notion independent of quantum physics. However, in some cases they can be used not only for detecting quantum entanglement, but for detecting genuine N-qubit entanglement [9]. Bell inequalities typically require measuring two variables at each qubit and computing an expression constructed as a sum of some N-qubit correlations. If the value of this expression is larger than a certain bound the system is N-qubit entangled. The drawback of this method is that it requires using very many *measurement settings*. As shown in Fig. 1(a), they need typically 2^N local measurement settings for detecting entanglement close to GHZ states.

Let us shortly explain what we understand by such a local measurement setting. Measuring a local setting $\{O^{(k)}\}$ with $k = 1,...,N$ consists of simultaneously performing the von Neumann measurements $O^{(k)}$ on the corresponding qubits, indexed by k. After repeating the measurements several times, the coincidence probabilities for the outcomes are collected. For N qubits there are 2^N different outcome probabilities. Given these probabilities it is possible to compute all the two point correlations $\langle O^{(k)}O^{(l)}\rangle$, three-point correlations $\langle O^{(k)}O^{(l)}O^{(m)}\rangle$, etc. Since all these correlation terms can be measured with one setting, the number of settings determines the experimental effort rather than the number of measured correlation terms. Obviously, with one local measurement setting it is not possible to detect entanglement. Thus, two measurement settings are the minimal effort needed for the detection of entanglement.

In this paper we explain some of the ideas of Ref. [10] on the detection of genuine multi-qubit entanglement. We will present entanglement conditions which require only *two* settings independent from the number of qubits. Since the number of measurement settings needed for existing methods increases exponentially with the number of qubits, our conditions provide a very effective way to detect entanglement. For instance, they need 2^{N-1} times less measurement settings than Bell inequalities. For a large number of qubits this difference is crucial — the new conditions do not improve things only quantitatively, they make detection possible when it would be unrealistic otherwise.

Our conditions will be presented in the form of *entanglement witnesses* [11]. These are operators which have a positive or zero expectation value for all separable states. Thus a negative expectation value signals the presence of entanglement. In constructing the entanglement witnesses for cluster states and GHZ states we use the *stabilizing operators* of these states. An observable S is a stabilizing operator of a state $|\Psi\rangle$ if it satisfies

$$S|\Psi\rangle = |\Psi\rangle. \tag{1}$$

In our case the operators S are tensor products of Pauli spin matrices. They can, therefore, easily be measured locally.

DETECTING GHZ STATES

Let us start with GHZ states. An N qubit GHZ state is defined as

$$|GHZ_N\rangle = \frac{1}{\sqrt{2}}(|0...0\rangle + |1...1\rangle). \tag{2}$$

As maximally entangled multi-qubit states, GHZ states are intensively studied [12] and have been realized in numerous experiments [6, 7]. The stabilizing operators of an N qubit GHZ state are

$$\begin{aligned} S_1^{(GHZ_N)} &:= \prod_{k=1}^{N} \sigma_x^{(k)}, \\ S_k^{(GHZ_N)} &:= \sigma_z^{(k-1)}\sigma_z^{(k)};\ k \in \{2,3,..,N\}. \end{aligned} \tag{3}$$

In fact, one can easily calculate that for these observables $S_k^{(GHZ_N)}|GHZ_N\rangle = |GHZ_N\rangle$ holds. Note that not only $S_k^{(GHZ_N)}$ stabilize the GHZ state, but any products of these operators does it as well. These operators form a commutative group and the $S_k^{(GHZ_N)}$ are the generators of the group.

One entanglement witness detecting genuine N-qubit entanglement close to GHZ states is given by

$$\mathscr{W}_{GHZ_N} := 3\cdot\mathbb{1} - 2\left[\frac{S_1^{(GHZ_N)}+\mathbb{1}}{2} + \prod_{k=2}^{N}\frac{S_k^{(GHZ_N)}+\mathbb{1}}{2}\right]. \tag{4}$$

The witness $\mathscr{W}_{GHZ_N}$ uses only two measurement settings, namely the ones in Fig. 1(b). The structure of $\mathscr{W}_{GHZ_N}$ given in Eq. (4) can be interpreted as follows. The two terms in the square brackets are two projectors. The first is a projector on the subspace for which $\langle S_1^{(GHZ_N)}\rangle = +1$. The second one is a projector on the subspace for which $\langle S_k^{(GHZ_N)}\rangle = +1$ for any $k \in \{2,3,...,N\}$. The GHZ state is the only state which is in both spaces, thus the mean value of $\mathscr{W}_{GHZ_N}$ is -1 only for this state. For any other state it is larger. In general, the more negative $\langle \mathscr{W}_{GHZ_N}\rangle_{|\Psi\rangle}$ is, the closer $|\Psi\rangle$ is, in some sense, to the GHZ state [13]. It is known that in the proximity of the GHZ state there are only states with genuine N-qubit entanglement, so the constant in Eq. (4) is chosen such that if $\langle \mathscr{W}_{GHZ_N}\rangle < 0$ then the state is in this neighborhood and is detected as entangled.

From the practical point of view it is very important to know, how large the neighborhood of the GHZ state is which is detected as entangled by the witness. This is usually characterized by the robustness to noise. Let us consider a GHZ state mixed with white noise

$$\rho(p_{noise}) := p_{noise}\cdot\frac{\mathbb{1}}{2^N} + (1-p_{noise})|GHZ_N\rangle\langle GHZ_N|. \tag{5}$$

The witness $\mathscr{W}_{GHZ_N}$ detects the state as entangled if $p_{noise} < 1(3-4/2^N)$. The bound on noise is explicitly shown in Table I. Our witness is quite robust — it tolerates at least 33% noise even for large N.

DETECTING CLUSTER STATES

Now let us turn to cluster states. These have recently raised a lot of interest both theoretically and experimentally. They can easily be created in a spin chain with Ising-type interaction [5] and have been realized in optical lattices of two-state atoms [8]. Remarkably, their entanglement is more persistent to noise than that of a GHZ state [5]. They play a central role in error correction [14], fault-tolerant quantum computation, cryptographic protocols such as secret sharing [15], and measurement-based quantum computation [16].

For three qubits the cluster state $|C_3\rangle$ is equivalent to a GHZ state up to local unitary transformations. For four qubits the state $|C_4\rangle$ can be transformed by some local unitaries into $|\phi\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$. For an arbitrary number of qubits it is more convenient to use a general definition via stabilizing operators than writing it out explicitly in some basis. The stabilizing operators of an N-qubit cluster state are

$$\begin{aligned} S_1^{(C_N)} &:= \sigma_x^{(1)}\sigma_z^{(2)}, \\ S_k^{(C_N)} &:= \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)}; k \in \{2,3,..,N-1\}, \\ S_N^{(C_N)} &:= \sigma_z^{(N-1)}\sigma_x^{(N)}. \end{aligned} \tag{6}$$

Given these stabilizing operators, the cluster state $|C_N\rangle$ is *defined* as the state fulfilling

$$S_i^{(C_N)}|C_N\rangle = |C_N\rangle. \tag{7}$$

TABLE 1. Noise tolerance for the GHZ state and the cluster state witnesses vs. number of qubits

N	2	3	4	5	6	7	8	9	10
$\vert GHZ_N\rangle$	0.50	0.40	0.36	0.35	0.34	0.34	0.34	0.33	0.33
$\vert C_N\rangle$	0.50	0.40	0.33	0.31	0.29	0.28	0.27	0.26	0.26

On can show that the cluster state is uniquely defined by these equations. Our witness for the detection of N-qubit entanglement around a cluster state is

$$\mathcal{W}_{C_N} \quad := \quad 3 \cdot \mathbb{1} - 2\left[\prod_{\text{even k}} \frac{S_k^{(C_N)} + \mathbb{1}}{2} + \prod_{\text{odd k}} \frac{S_k^{(C_N)} + \mathbb{1}}{2}\right]. \tag{8}$$

If the expectation value of $\mathcal{W}_{C_N}$ is negative then the system is genuine N-qubit entangled. Again, only two settings are needed. These are shown in Fig. 1(c). The witness tolerates at least 25% noise as shown in Table I. The structure of $\mathcal{W}_{C_N}$ is similar to that of $\mathcal{W}_{GHZ_N}$. In the square brackets there are two terms. The first term is a projector on the subspace for which $\langle S_k^{(C_N)}\rangle = +1$ for even k. The second term is a projector on the subspace for which $\langle S_k^{(C_N)}\rangle = +1$ for odd k.

SUMMARY

In summary, we have presented entanglement witnesses for detecting genuine multi-qubit entanglement close to GHZ and cluster states. These witnesses are easy to measure since they require only two measurement settings. For further details, especially for the proofs of the theorems presented here, please see Ref. [10]. This reference also describes how to generalize the results for graph states.

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REFERENCES

1. W. Dür, G. Vidal, and J. I. Cirac Phys. Rev. A **62**, 062314 (2000).
2. A. Acín *et al.*, Phys. Rev. Lett. **87**, 040401 (2001).
3. F. Verstraete *et al.*, Phys. Rev. A **65**, 052112 (2002).
4. D.M. Greenberger *et al.*, Am. J. Phys. **58**, 1131 (1990).
5. H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001).
6. J.-W. Pan *et al.*, Nature **403**, 515 (2000); Z. Zhao *et al.*, Phys. Rev. Lett. **91**, 180401 (2003); Z. Zhao *et al.*, Nature **430**, 54 (2004).
7. C.A. Sackett *et al.*, Nature **404**, 256 (2000).
8. O. Mandel *et al.*, Nature **425**, 937 (2003).
9. D. Collins *et al.*, Phys. Rev. Lett. **88**, 170405 (2002); N. Gisin, H. Bechmann-Pasquinucci, Phys. Lett A **246**, 1 (1998).
10. G. Tóth and O. Gühne, quant-ph/0405165.
11. M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996); B. M. Terhal, Phys. Lett. A **271**, 319 (2000); M. Lewenstein *et al.*, Phys. Rev. A **62**, 052310 (2000); D. Bruß *et al.*, J. Mod. Opt. **49**, 1399 (2002).
12. N.D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); M. Ardehali, Phys. Rev A **46**, 5375 (1992).
13. To be more specific, $\langle \mathcal{W}^{(GHZ_N)}\rangle_{|\Psi\rangle}$ can be used to find a lower bound on the overlap $|\langle GHZ_N|\Psi\rangle|^2$.
14. D. Gottesman, Phys. Rev. A **54**, 1862 (1996).
15. R. Cleve *et al.* Phys. Rev. Lett. **83**, 648 (1999).
16. R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).