Bell inequalities (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

Géza Tóth

Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
Donostia International Physics Center (DIPC), San Sebastián, Spain
IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
Wigner Research Centre for Physics, Budapest, Hungary

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Outline

What are Bell inequalities?

- Historically, the first inequalities showing that many-body quantum phenomena can lead to consequences very different from classical ones.

- Introduced by John Bell in 1964.

Outline

EPR paradox

of lanthanum is 7/2, hence the nuclear magnetic moment as determined by this analysis is 2.5 nuclear magnetons. This is in fair agreement with the value 2.8 nuclear magnetons determined from La III hyperfine structures by the writer and N. S. Grace.[9]

[9] M. F. Crawford and N. S. Grace, Phys. Rev. **47**, 536 (1935).

This investigation was carried out under the supervision of Professor G. Breit, and I wish to thank him for the invaluable advice and assistance so freely given. I also take this opportunity to acknowledge the award of a Fellowship by the Royal Society of Canada, and to thank the University of Wisconsin and the Department of Physics for the privilege of working here.

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Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

1.

ANY serious consideration of a physical theory must take into account the dis-

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counter-*

EPR paradox II

Paper by Einstein, Podolsky, and Rosen (EPR), Phys. Rev. 1935.

- The paper considered two particles in a singlet state

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- Let us call the two parties A and B (Alice and Bob).

- Some simple measurement scenarios are the following

Alice	Bob
$z = +1$	$z = -1$
$z = -1$	$z = +1$
$x = +1$	$z = \pm 1$

EPR paradox III

Questions:

- How does Bob's particle know, what Alice measured?
- The outcome is random in some cases. We should be able to determine the outcome of the measurement. Is not physics deterministic?
- Maybe, we just do not have enough information. There can be sofar unknown elements of reality that determine the measurement outcome.

Outline

Local hidden variable models

- Do the measured quantities correspond to an element of reality before the measurement? Let us assume that they do. (Reality)
- Assume that no faster than light communication is possible. (Locality).

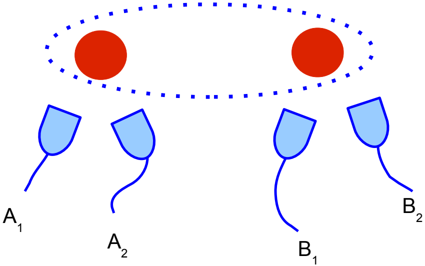


Figure:

Bipartite quantum system. We measure A_1 and A_2 at party A, and measure B_1 and B_2 at party B.

Local hidden variable models II

- Assume that we measure A_1 and A_2 at party A, and measure B_1 and B_2 at party B. Both A_k and B_k have ± 1 measurement results.
- A_k and B_k are quantum mechanically incompatible.
- Let us assume that all the four measurement outcomes exist before the measurement.
- The idea is that at each measurement k, there are $a_{1,k}, a_{2,k}, b_{1,k}, b_{2,k}$ available.
- We will show that quantum mechanics is not like that.

Local hidden variable models III

- We expect a measurement record like the following:

k	$a_{1,k}$	$a_{2,k}$	$b_{1,k}$	$b_{2,k}$
1	$+1$	-1	$+1$	$+1$
2	-1	$+1$	$+1$	-1
3	$+1$	$+1$	-1	$+1$
4	-1	-1	$+1$	-1
5	$+1$	$+1$	$+1$	-1
6	-1	-1	-1	$+1$
...	...	...	...	...

- Red color indicates the measured values. The other values we cannot check, we can only assume that they were there.

Local hidden variable models IV

- The correlations can be obtained as

$$\langle A_m B_n \rangle = \frac{1}{M} \sum_{k=1}^{M} a_{m,k} b_{n,k}.$$

- Here, k is the *hidden variable.* If I knew k, I could tell the outcome. It is hidden, but it is there somewhere.

Local hidden variable models V

- Usual formula, with λ as a hidden variable for probablities of outcomes for the discrete case

$$p(a_i^\alpha, b_j^\beta) = \int d\lambda \, p(\lambda)\mathcal{A}_\lambda(a_i^\alpha)\mathcal{B}_\lambda(b_j^\beta),$$

  Continuous case:

$$f(a_m, b_n) = \int f_{m,\lambda}(a_m) g_{n,\lambda}(b_n) d\lambda$$

  Here f's and $g's$ are probability density functions.

- In words: all two-variable probability distributions can be given as a sum of product distributions.

Outline

1. **Bell inequalities**
   - Motivation
   - A. EPR paradox
   - B. Local hidden variable models
   - C. The CHSH Bell inequality
   - D. Loopholes
     - Detection efficiency loophole
     - Locality loophole
   - E. Mermin's inequality

The CHSH Bell inequality

- Let us consider the following expression:

$$A_1 B_1 + A_2 B_1 + A_1 B_2 - A_2 B_2.$$

- Let us now substitute $+1$ or -1 to A_k and B_k. There are 16 combinations. We obtain

$$A_1 B_1 + A_2 B_1 + A_1 B_2 - A_2 B_2 \leq 2$$

- But, if we identify A with σ_x and B with σ_y, then there is a quantum state for which

$$\langle \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_x + \sigma_x \otimes \sigma_y - \sigma_y \otimes \sigma_y \rangle = 2\sqrt{2}.$$

This state is, apart from local transformations, the singlet $|01\rangle - |10\rangle$. How is this possible?

The CHSH Bell inequality II

- The real measurement record is the following:

k	$a_{1,k}$	$a_{2,k}$	$b_{1,k}$	$b_{2,k}$
1	$+1$	...	$+1$	...
2	-1	...	...	-1
3	...	$+1$	-1	...
4	-1	...	$+1$	...
5	...	$+1$	...	-1
6	-1	...	-1	...
...	...	...	...	...

- The correlations can be obtained as

$$\langle A_m B_n \rangle = \frac{1}{|\mathcal{M}_{m,n}|} \sum_{k \in \mathcal{M}_{m,n}} a_{m,k} b_{n,k},$$

  where $\mathcal{M}_{m,n}$ contains the indices corresponding to measuring A_m and B_n. This is the reason that correlations do not fit an LHV model.

Summary of Bell inequalities

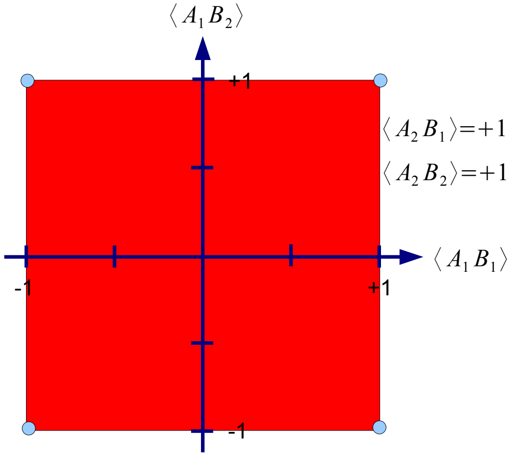
- Bell inequalities are made for bipartite (or multipartite systems).
- At each party, we measure one of two (or more) operators, such as σ_x and σ_y.
- Bell inequalities are inequalities with correlation terms that are constructed to exclude LHV models.
- They have the form
$$\langle B \rangle \leq C,$$
where B is the Bell operator and C is a constant.

Summary of Bell inequalities II

- *C* is the classical maximum. *C* is obtained from maximizing the operator *B* for all cases when we replace the operators with the measurement results. E.g., we replace σ_x with $+1$ or -1. In this way we obtain the maximum assuming
  - Reality: All outcomes of all measurement results exist before the measurements.
  - Locality: Alice does now know what Bob measures.
- If for a quantum state $|\Phi\rangle$ we have $\langle B\rangle_\Phi > C$ then we say that the Bell inequality is violated by the quantum state.
- We can also say that the measurement results cannot be described by an LHV model.

Summary of Bell inequalities III

- The points corresponding to correlations fulfilling Bell inequalities are within a polytope. Extreme points have correlations ±1.



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     - Detection efficiency loophole
     - Locality loophole
   - E. Mermin's inequality

Detection efficiency loophole

- Only very small part of the photons are detected by a detector. (The detector efficiency is typically much below 100%.)
- Maybe, only the statistics of the detected events violate the Bell inequalities.
- If we knew the statistics of all events, we would not get a Bell inequality violation.
- Typically problem with photons.

Locality loophole

- For each Bell inequality, at each party one of at least two operators is measured.
- If one party might know what is measured at the other party, some unknown mechanism could still mimic the violation of the Bell inequalities by communicating between the parties.
- Typically problem with trapped cold ions.

Outline

Mermin's inequality

For N qubits, the Mermin inequality is given by

$$\sum_{\pi}\langle X_1 X_2 X_3 X_4 X_5 \cdots X_N\rangle - \sum_{\pi}\langle Y_1 Y_2 X_3 X_4 X_5 \cdots X_N\rangle + \sum_{\pi}\langle Y_1 Y_2 Y_3 Y_4 X_5 \cdots X_N\rangle - \ldots + \ldots \leq L_{\text{Mermin}},$$

where $\sum_{\pi}$ represents the sum of all possible permutations of the particles that give distinct terms. $X_k, Y_k \in \{-1, +1\}$. L_{Mermin} is the maximum for local states. It is defined as

$$L_{\text{Mermin}} = \begin{cases} 2^{N/2} & \text{for even } N, \\ 2^{(N-1)/2} & \text{for odd } N. \end{cases}$$

- The quantum maximum is 2^{N-1} (all terms are $+1$).

Mermin's inequality II

- The state maximally violating the Mermin inequality is the GHZ state.
- The GHZ state is defined as
$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|00...00\rangle + |11..11\rangle\right).$$
- For the GHZ state we can identify X_k and Y_k with the Pauli spin matrices σ_x and σ_y.

Mermin's inequality III

- Example for $N = 3$ qubits. The Mermin inequality is given as

$$\langle X_1 X_2 X_3 \rangle - \langle Y_1 Y_2 X_3 \rangle - \langle Y_1 X_2 Y_3 \rangle - \langle X_1 Y_2 Y_3 \rangle \leq 2.$$

- Then, for the GHZ state we get 4 on the left-hand side, since for the GHZ states

$$\langle X_1 X_2 X_3 \rangle - \langle Y_1 Y_2 X_3 \rangle - \langle Y_1 X_2 Y_3 \rangle - \langle X_1 Y_2 Y_3 \rangle = 4,$$

since

$$\begin{aligned}
\langle \sigma_x \otimes \sigma_x \otimes \sigma_x \rangle &= +1, \\
\langle \sigma_y \otimes \sigma_y \otimes \sigma_x \rangle &= -1, \\
\langle \sigma_y \otimes \sigma_x \otimes \sigma_y \rangle &= -1, \\
\langle \sigma_x \otimes \sigma_y \otimes \sigma_y \rangle &= -1.
\end{aligned}$$

Recent loophole free experiments

- Lopphole free experiments are difficult, thus they have been carried out only recently.
- L. K. Shalm *et al.*, *Strong Loophole-Free Test of Local Realism*, Phys. Rev. Lett. 115, 250402 (2015).
- M. Giustina *et al., Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons*, Phys. Rev. Lett. 115, 250401 (2015).
- B. Hensen *et al.*, *Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres*, Nature (London) 526, 682 (2015).
- W. Rosenfeld, D. Burchardt, R. Garthoff, K. Redeker, N. Ortegel, M. Rau, and H. Weinfurter, *Event-Ready Bell Test Using Entangled Atoms Simultaneously Closing Detection and Locality Loopholes*, Phys. Rev. Lett. 119, 010402 (2017).

Photonic experiment without a locality loophole

- G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, *Violation of Bell's Inequality under Strict Einstein Locality Conditions*, Phys. Rev. Lett. 81, 5039 (1998).

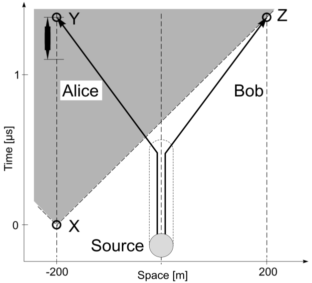

FIG. 1. Spacetime diagram of our Bell experiment. Selecting a random analyzer direction, setting the analyzer, and finally detecting a photon constitute the measurement process. This process on Alice's side must fully lie inside the shaded region which is invisible to Bob's during his own measurement. For our setup this means that the decision about the setting has to be made after point "*X*" if the corresponding photons are detected at spacetime points "*Y*" and "*Z*", respectively. In our experiment the measurement process (indicated by a short black bar) including the choice of a random number took less than only one-tenth of the maximum allowed time. The vertical parts of the kinked photon world lines emerging from the source represent the fiber coils at the source location, which are obviously irrelevant to the locality argument.

Photonic experiment without a locality loophole II

- In the photonic experiments, the horizontal/vertical (H/V) polarization of the photon encodes the logical 0 and 1, respectively.

- The state analyzed is

  entangled state $|\Psi\rangle = 1/\sqrt{2}\,(|H\rangle_1|V\rangle_2 + e^{i\varphi}|V\rangle_1|H\rangle_2)$, which we chose $\varphi = \pi$.

  That is, we have

  $$|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- The qubit can be measured with polarizing beam splitters (PBS) and detectors. On the following figure, PBS is called "Polarizer".

Photonic experiment without a locality loophole III

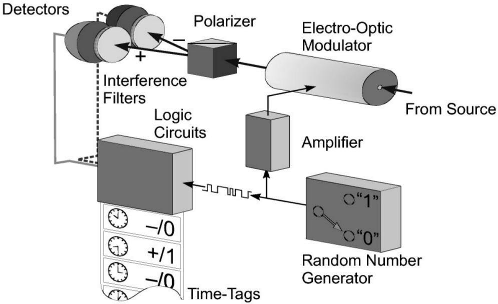

FIG. 2. One of the two observer stations. A random number generator is driving the electro-optic modulator. Silicon avalanche photodiodes are used as detectors. A "time tag" is stored for each detected photon together with the corresponding random number "0" or "1" and the code for the detector "+" or "−" corresponding to the two outputs of the polarizer.

Photonic experiment without a locality loophole IV

- The experiment tests the CHSH inequality:

coincidence rates. In a rather general form the CHSH inequality reads

$$S(\alpha, \alpha', \beta, \beta') = |E(\alpha, \beta) - E(\alpha', \beta)| + |E(\alpha, \beta') + E(\alpha', \beta')| \leq 2 . \quad (1)$$

- The correlations are defined as follows:

$E(\alpha, \beta)$ of the correlation between Alice's and Bob's local results is $E(\alpha, \beta) = [C_{++}(\alpha, \beta) + C_{--}(\alpha, \beta) - C_{+-}(\alpha, \beta) - C_{-+}(\alpha, \beta)]/N$, where N is the sum of all coincidence rates. In a rather general form the CHSH

This correlation term is like $\langle \sigma_z \otimes \sigma_z \rangle$ or $\langle \sigma_{\vec{n}_1} \otimes \sigma_{\vec{n}_2} \rangle$, where $\vec{n}_k$ are some spin directions.

Photonic experiment without a locality loophole V

Summary of the experiment:

- CHSH experiment with two photons, without the locality loophole.
- Note that the detection loophole has not been solved that time.