Entanglement measures
(How much is it entangled?)
(Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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Outline

1. **Entanglement measures (How much is it entangled?)**
    - Motivation
    - A. General quantum operation
    - B. Local operations and classical communication (LOCC)
    - C. Entanglement of formation
    - D. Concurrence
    - E. Entanglement of distillation
    - F. Bound entanglement
    - G. Requirements for entanglement measures
    - H. Negativity

Entanglement measures

- After detecting entanglement, we have to ask how entangled the state is.
- It will turn out that entanglement is a resource.

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General quantum operation

- The general quantum operation is defined as

  $$\varrho' = \sum_k E_k \varrho E_k^\dagger$$

  with

  $$\sum_k E_k^\dagger E_k = 1.$$

- E_k are Kraus operators.
- Generalized measurements, POVM (positive operator-valued measure).
- Special case: von Neumann measurements, when E_k are pairwise orthogonal projectors.
- Naimark's dilation theorem:
  general operation=
  von Neumann measurement on system+ancilla.

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Local operations and classical communication (LOCC)

- LOCC are
    - local unitaries,
    - local von Neumann or POVM measurements,
    - local unitaries or measurements conditioned on measurement outcomes on the other party.
- Mathematical description of LOCC. Separable operations are a somewhat larger set, however, this set can easily be described.

$$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

with

$$\sum_k \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger \left(E_k^{(1)} \otimes E_k^{(2)} \right) = 1.$$

Local operations and classical communication (LOCC) II

- Stochastic Local Operations and Classical Communication (SLOCC):

  $$|\Psi\rangle' \leftarrow E_k^{(1)} \otimes E_k^{(2)} |\Psi\rangle$$

  It happens with some probability, not deterministic.
- LOCC cannot create entanglement. Separable states remain separable under LOCC.
- LOCC can create correlations.

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Entropy of entanglement

- The von Neumann entropy is defined as

$$S(\varrho) = -\mathrm{Tr}(\varrho \log_2 \varrho).$$

- It can be written with the eigenvalues of the density matrix as

$$S(\varrho) = -\sum_{k=1}^{d} \lambda_k \log_2 \lambda_k.$$

- For a pure state we have $\lambda_k = \{1, 0, 0, ..., 0\}$, and thus it is zero.
- Its maximal is for the completely mixed state for which $\lambda_k = \{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, ..., \frac{1}{d}\}$, and its value is $\log_2 d$.
- For a bipartite pure state, the entropy of entanglement is

$$E_E(|\Psi\rangle) = \mathrm{S}(\mathrm{Tr}_1(|\Psi\rangle\langle\Psi|)).$$

That is, it is the von Neumann entropy of the reduced state is an entaglement measure.

Entropy of entanglement II

- Comments
    - It is one for two-qubit singlet states.
    - It is zero for product states.
    - It is invariant under $U_1 \otimes U_2$.

Entanglement of formation

- For mixed states, the entanglement of formation is the convex roof of the von Neumann entropy of the reduced state.

$$E_F = \min_{|\Psi_k\rangle, p_k} \sum_k p_k E_E(|\Psi_k\rangle),$$

- The optimization is over all decompositions of the state of the type

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- E_F tells us, in the asymptotic limit, how many singlets we need to create the state.
- Is it easy to compute? No. For 2×2 systems, there is an explicit formula with the concurrence. For larger systems, there is not a general method.

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Entanglement of formation

- For two qubits, E_F can be calculated explicitly (Wootters, 1997).
- Special case: for pure states the concurrence is

$$C(|\Psi\rangle) = |\langle\Psi|\tilde{\Psi}\rangle| = 2|a_{11}a_{22} - a_{12}a_{21}|,$$

where

$$|\Psi\rangle = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

- It is related to the linear entropy of the reduced state.

$$C = \sqrt{2(1 - \mathrm{Tr}(\rho_{\mathrm{red}}^2)}, \tag{1}$$

where

$$\rho_{\mathrm{red}} = \mathrm{Tr}_2(|\Psi\rangle\langle\Psi|). \tag{2}$$

Entanglement of formation II

- Now we have to compute E_F from C.
- We also nee that

$$\epsilon(c) = H_2\left(\frac{1+\sqrt{1-c^2}}{2}\right).$$

Here

$$H_2 = -x\log_2 x - (1-x)\log_2(1-x).$$

- Then, E_F can be obtained as

$$E_F(\varrho) = \epsilon(C(\varrho)).$$

Entanglement of formation III

- For mixed states, the concurrence is defined as

  $$C(\varrho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

  where λ_k's are, in a decreasing order, the eigenvalues of

  $$R = \sqrt{\sqrt{\varrho}\tilde{\varrho}\sqrt{\varrho}},$$

  and

  $$\tilde{\varrho} = (\sigma_y \otimes \sigma_y)\varrho^*(\sigma_y \otimes \sigma_y).$$

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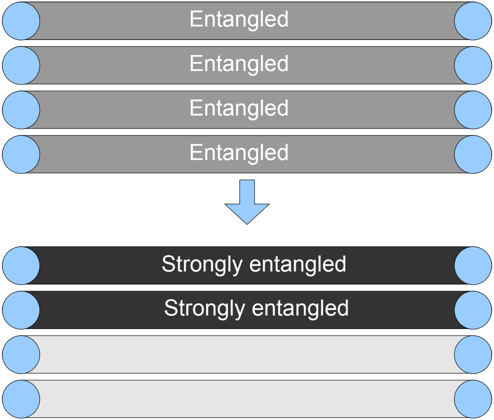
Entanglement of distillation

- E_D tells us, how many singlets we can obtain from the state with LOCC. In general,
$$E_F \geq E_D.$$
- Note that local operation and classical communication means that we have several copies and we can act on the copies locally.

Entanglement of distillation II



Entangled

Entangled

Entangled

Entangled

Strongly entangled

Strongly entangled

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Bound entanglement

- There are states that need entangled particles to be created, but singlets cannot be distilled from them.
- All PPT entangled states are like that. (That is, all entangled states that are not detected by the Peres-Horodecki criterion.)

Bound entanglement II

- Next, we will prove this. First we show that PPT state remain PPT under LOCC. Under LOCC we have

  $$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

  We also have

  $$(\varrho')^{T2} = \sum_k E_k^{(1)} \otimes ((E_k^{(2)})^\dagger)^T \varrho^{T2} (E_k^{(1)})^\dagger \otimes (E_k^{(2)})^T$$

  Here we used that $(AB)^T = B^T A^T$ and $A^\dagger = (A^*)^T$.

- We can see that if $\varrho^{T2} \geq 0$ then $(\varrho')^{T2} \geq 0$. Thus the PPT states remain PPT under LOCC.

  R., P., M., and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009). (Click on the link above, see "G. Bound entanglement - when distillability fails" on page 44.)

Bound entanglement III

- Let us again remember the flip operator

  $$F|k\rangle|l\rangle = |l\rangle|k\rangle$$

  It has eigenvalues ±1.

- The maximally entangled state

  $$|\Psi_{\rm me}\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k\rangle|k\rangle.$$

Bound entanglement IV

- We can show that

$$\begin{aligned} |\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}| &= \frac{1}{d}\sum_{k,l}^{d} |k\rangle\langle l| \otimes |k\rangle\langle l|, \\ |\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|^{T1} &= \frac{1}{d}\sum_{k,l}^{d} |k\rangle\langle l| \otimes |l\rangle\langle k| \equiv \frac{F}{d}. \end{aligned}$$

- Now we show that PPT states have a small overlap with the maximally entangled state. For PPT states, the fidelity with respect to the maximally entangled state is

$$\mathrm{Tr}(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|\varrho) = \mathrm{Tr}(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|^{T1}\varrho^{T1}) = \frac{1}{d}\mathrm{Tr}(F\varrho^{T1}) \leq \frac{1}{d},$$

  since $\varrho^{T1} \geq 0$ and F has ± 1 eigenvalues.

Bound entanglement IV

- Thus, PPT states have a small fidelity with respect to the maximally entangled state. Even LOCC operations cannot increase this.

- A simple product state can reach $1/d$

$$\mathrm{Tr}(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}||11\rangle\langle 11|) = \frac{1}{d}.$$

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Requirements for entanglement measures

1. To each density matrix it assigns a nonnegative number. Typically, the maximally entangled state has $\log d$.
2. $E(\varrho) = 0$ for separable states.
3. E does not increase on average under LOCC.

$$E(\varrho) \leq \sum_k p_k E\left(\frac{A_k \varrho A_k^\dagger}{\mathrm{Tr}(A_k \varrho A_k^\dagger)}\right). \tag{3}$$

4. For pure states, it has the same value as the entangement entropy.

- Entanglement monotone: 1,2,3.
- Entanglement mesure: 1,2,4 and does not increase under deterministic LOCC, i.e.,

$$E(\varrho') \leq E(\varrho); \quad \varrho' = \sum_k A_k \varrho A_k^\dagger \quad \text{(POVM)}. \tag{4}$$

M. B. Plenio and S. Virmani, eprint arXiv:quant-ph/0504163 (2005).

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Trace norm

- Let us consider the singular decomposition of a matrix

  $$A = U\Sigma V^{\dagger}, \tag{5}$$

  where

  $$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_d) \tag{6}$$

  and $\sigma_k > 0$.

- Then the trace norm is

  $$\|A\|_1 = \mathrm{Tr}\left(\sqrt{AA^{\dagger}}\right) = \sum_k \sigma_k. \tag{7}$$

- The Hilbert-Schmidt norm is

  $$\|A\|_2 = \mathrm{Tr}\left(AA^{\dagger}\right) = \sum_k \sigma_k^2. \tag{8}$$

Negativity

- Example for a monotone: negativity

$$N(\varrho) = \frac{\|\varrho^{\mathrm{T1}}\| - 1}{2}.$$

Trace norm=sum of singular values.

- For Hermitian matrices, it is the same as sum of eigenvalues.

$$N(\varrho) = \frac{\sum_k |\lambda_k| - 1}{2}.$$

- Note that $\sum_k \lambda_k = 1$. Then, assume that the first M eigenvalues are negative, the rest is positive. We get

$$N(\varrho) = \frac{\sum_{k=1}^{M} -\lambda_k + \sum_{k=M+1}^{d} \lambda_k - \sum_k \lambda_k}{2}.$$

Negativity II

- Hence,

$$N(\varrho) = \sum_{k=1}^{M} |\lambda_k|.$$

  That is, the absolute value of the sum of the negative eigenvalues of the partial transpose.

- Clearly, it is zero for PPT states. Thus, it is zero for all separable states.

- Not as meaningful as the Entanglement of Formation, but can be calculated on any system sizes.

- It fulfills certain conditions on how it changes under LOCC. It does not increase under deterministic LOCC.