Entanglement measures
(How much is it entangled?)
(Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

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Outline

1. **Entanglement measures (How much is it entangled?)**
   - Motivation
   - A. General quantum operation
   - B. Local operations and classical communication (LOCC)
   - C. Entanglement of formation
   - D. Concurrence
   - E. Entanglement of distillation
   - F. Bound entanglement
   - G. Negativity

Entanglement measures

- After detecting entanglement, we have to ask how entangled the state is.
- It will turn out that entanglement is a resource.

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General quantum operation

- The general quantum operation is defined as

$$\varrho' = \sum_k E_k \varrho E_k^\dagger$$

  with

$$\sum_k E_k^\dagger E_k = 1.$$

- E_k are Kraus operators.
- Generalized measurements, POVM (positive operator-valued measure).
- Special case: von Neumann measurements, when E_k are pairwise orthogonal projectors.
- Naimark's dilation theorem:
  general operation=
  von Neumann measurement on system+ancilla.

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Local operations and classical communication (LOCC)

- LOCC are
  - local unitaries,
  - local von Neumann or POVM measurements,
  - local unitaries or measurements conditioned on measurement outcomes on the other party.

- Mathematical description of LOCC. Separable operations are a somewhat larger set, however, this set can easily be described.

$$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

with

$$\sum_k \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger \left(E_k^{(1)} \otimes E_k^{(2)} \right) = 1.$$

Local operations and classical communication (LOCC) II

- Stochastic Local Operations and Classical Communication (SLOCC):

  $$|\Psi\rangle' \leftarrow E_k^{(1)} \otimes E_k^{(2)} |\Psi\rangle$$

  It happens with some probability, not deterministic.
- LOCC cannot create entanglement. Separable states remain separable under LOCC.
- LOCC can create correlations.

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Entropy of entanglement

- The von Neumann entropy is defined as

$$S(\varrho) = -\mathrm{Tr}(\varrho \log_2 \varrho).$$

- It can be written with the eigenvalues of the density matrix as

$$S(\varrho) = -\sum_{k=1}^{d} \lambda_k \log_2 \lambda_k.$$

- For a pure state we have $\lambda_k = \{1, 0, 0, ..., 0\}$, and thus it is zero.
- Its maximal is for the completely mixed state for which $\lambda_k = \{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, ..., \frac{1}{d}\}$, and its value is $\log_2 d$.
- For a bipartite pure state, the entropy of entanglement is

$$E_E(|\Psi\rangle) = \mathrm{S}(\mathrm{Tr}_1(|\Psi\rangle\langle\Psi|)).$$

  That is, it is the von Neumann entropy of the reduced state is an entaglement measure.

Entropy of entanglement II

- Comments
  - It is one for two-qubit singlet states.
  - It is zero for product states.
  - It is invariant under $U_1 \otimes U_2$.

Entanglement of formation

- For mixed states, the entanglement of formation is the convex roof of the von Neumann entropy of the reduced state.

$$E_F = \min_{|\Psi_k\rangle, p_k} \sum_k p_k E_E(|\Psi_k\rangle),$$

- The optimization is over all decompositions of the state of the type

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- E_F tells us, in the asymptotic limit, how many singlets we need to create the state.
- Is it easy to compute? No. For 2×2 systems, there is an explicit formula with the concurrence. For larger systems, there is not a general method.

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Entanglement of formation

- For two qubits, E_F can be calculated explicitly (Wootters, 1997).
- Special case: for pure states the concurrence is

$$C(|\Psi\rangle) = |\langle\Psi|\tilde{\Psi}\rangle| = 2|a_{11}a_{22} - a_{12}a_{21}|,$$

where

$$|\Psi\rangle = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

- It is related to the linear entropy of the reduced state.

$$C = \sqrt{2(1 - \mathrm{Tr}(\rho_{\mathrm{red}}^2)}, \tag{1}$$

where

$$\rho_{\mathrm{red}} = \mathrm{Tr}_2(|\Psi\rangle\langle\Psi|). \tag{2}$$

Entanglement of formation II

- Now we have to compute E_F from C.
- We also nee that

$$\epsilon(c) = H_2\left(\frac{1+\sqrt{1-c^2}}{2}\right).$$

  Here

$$H_2 = -x\log_2 x - (1-x)\log_2(1-x).$$

- Then, E_F can be obtained as

$$E_F(\varrho) = \epsilon(C(\varrho)).$$

Entanglement of formation III

- For mixed states, the concurrence is defined as

$$C(\varrho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where λ_k's are, in a decreasing order, the eigenvalues of

$$R = \sqrt{\sqrt{\varrho}\tilde{\varrho}\sqrt{\varrho}},$$

and

$$\tilde{\varrho} = (\sigma_y \otimes \sigma_y)\varrho^*(\sigma_y \otimes \sigma_y).$$

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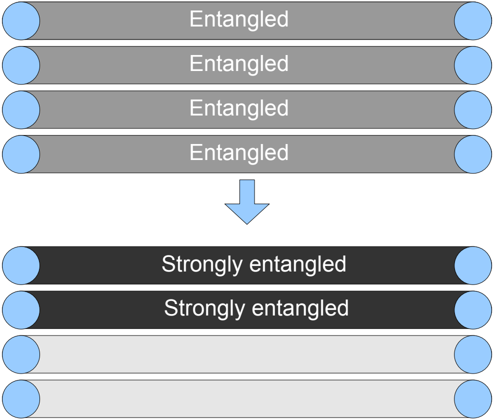
Entanglement of distillation

- E_D tells us, how many singlets we can obtain from the state with LOCC. In general,
$$E_F \geq E_D.$$
- Note that local operation and classical communication means that we have several copies and we can act on the copies locally.

Entanglement of distillation II



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Bound entanglement

- There are states that need entangled particles to be created, but singlets cannot be distilled from them.
- All PPT entangled states are like that. (That is, all entangled states that are not detected by the Peres-Horodecki criterion.)

Bound entanglement II

- Next, we will prove this. First we show that PPT state remain PPT under LOCC. Under LOCC we have

$$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho \left(E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

We also have

$$(\varrho')^{T2} = \sum_k E_k^{(1)} \otimes ((E_k^{(2)})^\dagger)^T \varrho^{T2} (E_k^{(1)})^\dagger \otimes (E_k^{(2)})^T$$

Here we used that $(AB)^T = B^T A^T$ and $A^\dagger = (A^*)^T$.

- We can see that if $\varrho^{T2} \geq 0$ then $(\varrho')^{T2} \geq 0$. Thus the PPT states remain PPT under LOCC.

R., P., M., and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
(Click on the link above, see "G. Bound entanglement - when distillability fails" on page 44.)

Bound entanglement III

- Let us again remember the flip operator

$$F|k\rangle|l\rangle = |l\rangle|k\rangle$$

It has eigenvalues ± 1.

- The maximally entangled state

$$|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle|k\rangle.$$

Bound entanglement IV

- We can show that

$$
\begin{aligned}
|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}| &= \frac{1}{d}\sum_{k,l}^{d}|k\rangle\langle l|\otimes|k\rangle\langle l|, \\
|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|^{T1} &= \frac{1}{d}\sum_{k,l}^{d}|k\rangle\langle l|\otimes|l\rangle\langle k| \equiv \frac{F}{d},
\end{aligned}
$$

  where $|l\rangle$ are real orthonormal vectors.

- Now we show that PPT states have a small overlap with the maximally entangled state. For PPT states, the fidelity with respect to the maximally entangled state is

$$
\mathrm{Tr}(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|\varrho) = \mathrm{Tr}(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|^{T1}\varrho^{T1}) = \frac{1}{d}\mathrm{Tr}(F\varrho^{T1}) \le \frac{1}{d},
$$

  since $\varrho^{T1} \ge 0$ and F has ± 1 eigenvalues.

Bound entanglement IV

- Thus, PPT states have a small fidelity with respect to the maximally entangled state. Even LOCC operations cannot increase this.
- A simple product state can reach $1/d$

$$\mathrm{Tr}(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}||11\rangle\langle 11|) = \frac{1}{d}.$$

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Trace norm

- Let us consider the singular decomposition of a matrix

$$A = U\Sigma V^{\dagger},$$

where

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, ..., \sigma_d)$$

and $\sigma_k > 0$.

- Then the trace norm is

$$||A||_1 = \mathrm{Tr}\left(\sqrt{AA^{\dagger}}\right) = \sum_k \sigma_k.$$

- The Hilbert-Schmidt norm is

$$||A||_2 = \mathrm{Tr}\left(AA^{\dagger}\right) = \sum_k \sigma_k^2.$$

Negativity

- Example for a monotone: negativity

$$N(\varrho) = \frac{\|\varrho^{\mathrm{T1}}\| - 1}{2}.$$

  Trace norm=sum of singular values.

- For Hermitian matrices, it is the same as sum of eigenvalues.

$$N(\varrho) = \frac{\sum_k |\lambda_k| - 1}{2}.$$

- Note that $\sum_k \lambda_k = 1$. Then, assume that the first M eigenvalues are negative, the rest is positive. We get

$$N(\varrho) = \frac{\sum_{k=1}^{M} -\lambda_k + \sum_{k=M+1}^{d} \lambda_k - \sum_k \lambda_k}{2}.$$

Negativity II

- Hence,
$$N(\varrho) = \sum_{k=1}^{M} |\lambda_k|.$$
That is, the absolute value of the sum of the negative eigenvalues of the partial transpose.

- Clearly, it is zero for PPT states. Thus, it is zero for all separable states.

- Not as meaningful as the Entanglement of Formation, but can be calculated on any system sizes.

- It fulfills certain conditions on how it changes under LOCC. It does not increase under deterministic LOCC.