General characteristics of multi-partite quantum systems (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

Géza Tóth

Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
Donostia International Physics Center (DIPC), San Sebastián, Spain
IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
Wigner Research Centre for Physics, Budapest, Hungary

UPV/EHU, Leioa
21 January, 2026

Outline

- Geometry of quantum states
  - A single qubit
  - A single qudit (qunit):d-dimensional systems

Bloch vector

- For a single qubit, **the density matrix has three real parameters.** It can be written as

$$\varrho = \frac{1}{2}\left(\mathbb{1} + \sum_{l=x,y,z} v_l \sigma_l\right), \tag{1}$$

  where σ_l are the Pauli spin matrices.

- Using $\mathrm{Tr}(\sigma_k \sigma_l) = 2\delta_{kl}$, we can write

$$\mathrm{Tr}(\varrho^2) = \frac{1}{2} + \frac{1}{2}\sum_{l=x,y,z} v_l^2. \tag{2}$$

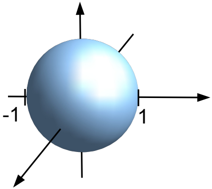
  That is, the Bloch vector has a maximal length for pure states.

Bloch vector II

- From $\mathrm{Tr}(\varrho^2) \leq 1$, the condition for being physical is Eq. (**??**), which is equvalent to
$$\sum_{l=x,y,z} |v_l|^2 \leq 1. \tag{3}$$
The three-element vector is called the Bloch vector.

Bloch vector III

- Let us identify the points in (v_x, v_y, v_z) corresponding to physical states. They are in a ball.
- The pure states are on the surface.
- Mixed states are inside the Ball. This is because $\mathrm{Tr}(\varrho^2)$ is directly related to the length of the Bloch vector.
- The $|0\rangle$ and $|1\rangle$ correspond to the North and South Pole.
- $|0\rangle + \exp(-i\phi)|1\rangle$ correspond to points on the equator.



Set of physical quantum states for a single qubit. The axes correspond to v_l for $l = x, y, z$. Pure states correspond to points on the surface, mixed states correspond to internal points.

A single qudit (qunit): d-dimensional systems

- For higher dimensional systems the picture is much more complicated. Let us consider qudits with dimension d.
- Similarly to the case before, **a $d \times d$ Hermitian matrix with a unit trace has $d^2 - 1$ degrees of freedom.**
- Hence, we can write a density matrix as a linear combination of $d^2 - 1$ SU(d) generators as

$$\varrho = \frac{1}{d}\mathbb{1} + \frac{1}{2}\sum_{l=1}^{d^2-1} v_l g_l. \tag{4}$$

Here,

$$\mathrm{Tr}(g_k g_l) = 2\delta_{kl}. \tag{5}$$

(Like for the Pauli matrices. Thus, we have something like the generalized Pauli matrices. $d = 3$: Gell-Mann matrices.)

A single qudit (qunit): d-dimensional systems

- For higher dimensional systems the picture is much more complicated. Let us consider qudits with dimension d.

- Similarly to the case before, **a $d \times d$ Hermitian matrix with a unit trace has $d^2 - 1$ degrees of freedom.** Hence, we can write a density matrix as a linear combination of $d^2 - 1$ SU(d) generators as

$$\varrho = \frac{1}{d}\mathbb{1} + \frac{1}{2}\sum_{l=1}^{d^2-1} v_l g_l. \tag{6}$$

  Here,

$$\mathrm{Tr}(g_k g_l) = 2\delta_{kl}. \tag{7}$$

- Like for the Pauli matrices. Thus, we have something like the generalized Pauli matrices. $d = 3$: for instance, Gell-Mann matrices.

A single qudit (qunit): d-dimensional systems II

- Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

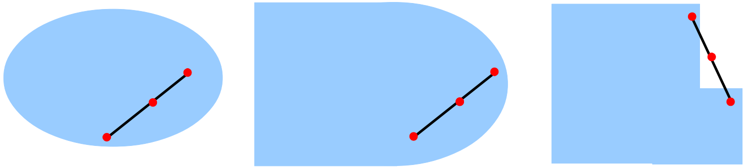
There are other possibilities: J. Lawrence, quant-ph/0403095.

A single qudit (qunit): d-dimensional systems III

- Let us again look at the points $(v_1, v_2, ..., v_{d^2-1})$ corresponding to physical states.
- First note that the set of convex. This is because mixing two physical states ϱ_1 and ϱ_2, we always get a physical state

$$\varrho = p\varrho_1 + (1-p)\varrho_2. \tag{8}$$

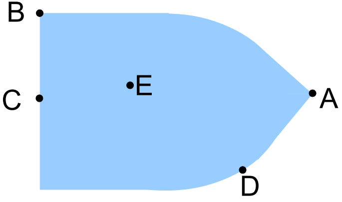
A single qudit (qunit): d-dimensional systems IV



Two convex objects and one that is not convex.

A single qudit (qunit): d-dimensional systems V

- On the next figure we will show the set of quantum states.
- The cooridnate axis could be the v_l, for example.
- Inside the set there are the density matrices with full rank.
- On the boundary there are the states with less than full rank, such as for example rank-1 states, which are pure states.



Set of physical quantum states. Note that the set is convex.
A,B,D: rank-1 states. C: rank-2 state. E: full rank states.

A single qudit (qunit): d-dimensional systems VI

- **Observation.** The following inequality is true

$$\lambda_{\min}(A+B) \geq \lambda_{\min}(A) + \lambda_{\min}(B). \tag{9}$$

Proof. Let us consider that for a Hermitian matrix X we have

$$\lambda_{\min}(X) = \min_{\psi} \langle\psi|X|\psi\rangle. \tag{10}$$

Then, for A and B Hermitian matrices we have

$$\begin{aligned} \lambda_{\min}(A+B) &= \min_{\psi}\langle\psi|A+B|\psi\rangle \geq \min_{\psi}\langle\psi|A|\psi\rangle + \min_{\psi}\langle\psi|B|\psi\rangle \\ &= \lambda_{\min}(A) + \lambda_{\min}(B). \end{aligned} \tag{11}$$

□

We can prove similarly that

$$\lambda_{\max}(A+B) \leq \lambda_{\max}(A) + \lambda_{\max}(B). \tag{12}$$

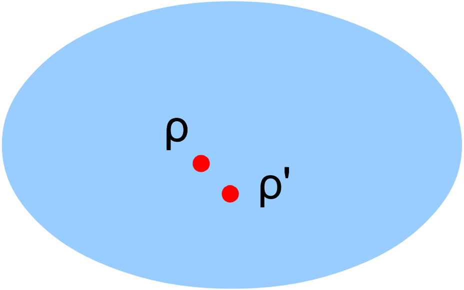
Full rank states

- Using this, we can say the following.
- **Observation.** Full-rank states are inside the set.
  Proof. If the state is full rank, it means that for some small ϵ

$$\varrho' = \varrho + \epsilon H \tag{13}$$

  is also physical, where H is a trace 0 Hermitian matrix. Why is that? See also the next figure.

Full rank states II



We take an internal state ϱ and consider the states ϱ' in its neighborhood.

Full rank states III

- It is physical since
  1. Trace is 1.
  2. Hermitian.
  3. Full rank means that

$$\lambda_{\min}(\varrho) > 0, \qquad \lambda_{\max}(\varrho) < 1. \tag{14}$$

Eigenvalues are nonzero for small $|\epsilon|$. This is because

$$\lambda_{\max}(\varrho) + \lambda_{\max}(\epsilon H) \geq \lambda_k(\varrho') \geq \lambda_{\min}(\varrho) + \lambda_{\min}(\epsilon H). \tag{15}$$

Here we have

$$\lambda_{\min}(\epsilon H) = \begin{cases} +\epsilon\lambda_{\min}(H), & \text{if } \epsilon \geq 0, \\ -|\epsilon|\lambda_{\max}(H), & \text{if } \epsilon < 0. \end{cases} \tag{16}$$

Similar statement holds for $\lambda_{\max}(\epsilon H)$. $\square$

Non-full-rank states

- **Observation.** Non-full-rank states are on the surface of the set.
- *Proof.* If the state is not full rank, then it has zero eigenvalues. Thus, there is an H such that ϱ' is aphisical for any $\epsilon > 0$ or any $\epsilon < 0$.
- To be more explicit, let us write

$$\varrho = UDU^\dagger, \tag{17}$$

such that D contains the eigenvalues. Here,

$$D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_d), \tag{18}$$

and the eigenvectors are

$$U = [|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle, ..., |\Psi_d\rangle]. \tag{19}$$

Non-full-rank states II

- Assume that $\lambda_d = 0$. Then,

$$\varrho' = \varrho + \epsilon(|\Psi_d\rangle\langle\Psi_d| - \mathbb{1}/d) \tag{20}$$

  has a negative eigenvalue for any $\epsilon < 0$. The Identity is needed to make the expression zero-trace.

- This is because the eigenvalues of this matrix are

$$D' = \mathrm{diag}(\lambda_1 - \epsilon/d, \lambda_2 - \epsilon/d, \lambda_3 - \epsilon/d, ..., \lambda_d + \epsilon(1 - 1/d)). \tag{21}$$

□