Interesting quantum states (Lecture of the Quantum Information class of the Master in Quantum Science and Technology)

Géza Tóth

Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
Donostia International Physics Center (DIPC), San Sebastián, Spain
IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
Wigner Research Centre for Physics, Budapest, Hungary

UPV/EHU, Leioa
26 and 28 January, 2021

Outline

Which quantum states are interesting?

- We have infinite possibilities to pick a quantum state in a multi-qubit system.
- We would like to find useful ones or states that have interesting symmetries.

Outline

1. **Interesting quantum states**
   - Motivation
   - A. Single particle states
   - B. Bipartite singlet state
   - C. Werner states
   - D. Isotropic states
   - E. Schrödinger cat states
   - F. Greenberger-Horne-Zeilinger (GHZ) state
   - G. W state
   - H. Symmetric Dicke states

Single particle states

- Pure states. The von Neumann entropy $S = 0$.
- Completely mixed state

$$\varrho_{\rm cm} = \frac{1}{d}\sum_{k=1}^{d} |k\rangle\langle k|.$$

The von Neumann entropy $S = \log d$, maximal.

Outline

Bipartite singlet state

- The two-qubit singlet state looks like

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- We get the same form after any basis transformation (if we transform the bases of the two qubits in the same way).This can be seen as follows. Let us choose two vectors as

$$\begin{aligned} |v\rangle &= \alpha|0\rangle + \beta|1\rangle, \\ |v_\perp\rangle &= \beta^*|0\rangle - \alpha^*|1\rangle. \end{aligned}$$

  Clearly,

$$\langle v|v_\perp\rangle = 0,$$

  Then, simple algebra yields

$$\frac{1}{\sqrt{2}}(|v\rangle \otimes |v_\perp\rangle - |v_\perp\rangle \otimes |v\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

  This is true for any $|v\rangle$ and $|v_\perp\rangle$.

Bipartite singlet state II

- Due to the independence from the choice of the local basis, it is invariant under a transformation of the type $U \otimes U$, apart from a global phase ϕ.

$$U \otimes U|\Psi_{\text{singlet}}\rangle = |\Psi_{\text{singlet}}\rangle \exp(-i\phi).$$

We can also say that

$$U \otimes U|\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}|(U \otimes U)^{\dagger} = |\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}|.$$

Hence,

$$U \otimes U|\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}| = |\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}|U \otimes U.$$

Thus, the density matrices of such states will commute with all $U \otimes U$.

Bipartite singlet state III

- Let us consider some operators of the form

$$\sigma_{\vec{n}} = \sum_{l=x,y,z} n_l \sigma_l$$

  where $|\vec{n}| = 1$. For $\vec{n} = (1,0,0)$, $\sigma_{\vec{n}} = \sigma_x$. For $\vec{n} = (0,1,0)$, $\sigma_{\vec{n}} = \sigma_y$, and in general it is a generalization of the Pauli spin matrices to an arbitrary direction.

- Such operators all have eigenvalues ± 1. If you measure $\sigma_{\vec{n}}$ on party A and get a result, then if you also measure it on party B, you will get the opposite result. This is true for every $\sigma_{\vec{n}}$.

- This can be used in quantum communication to establish a bit sequence that is known only by Alice and Bob and by nobody else.

Bipartite singlet state IV

- Why is it called a singlet? Remember the theory of angular momentum, triplet and singlet subspace.
- Alternatively, in quantum information, the maximally entangled state
$$|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$
can also be called singlet.
- A generalization for higher dimensions is the maximally entangled state
$$|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|kk\rangle.$$
- For the maximally entangled state, the reduced state is the completely mixed state
$$\mathrm{Tr}_A(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|) = \mathrm{Tr}_B(|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}|) = \frac{\mathbb{1}}{d}.$$
Thus, if we have access only to one of the two subsystems, we know nothing.

Outline

Flip operator

- Flip operator
$$F|\Psi\rangle|\Phi\rangle = |\Phi\rangle|\Psi\rangle.$$
- For two-qubits
$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
.
$$F = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z).$$
.
- Eigenvalues of F : -1,1,1,1 (-1 for antisymmetric states, +1 for symmetric states).

Flip operator II

- For a comparison, remember that for the projector to the singlet state we have

$$|\Psi_{\mathrm{singlet}}\rangle\langle\Psi_{\mathrm{singlet}}| = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z).$$

- For the projector to the maximally entangled state we have

$$|\Psi_{\mathrm{me}}\rangle\langle\Psi_{\mathrm{me}}| = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z).$$

Werner states

- **Definition.** Werner states are states that are invariant under a transformation of the type $U \otimes U$ (Werner, 1989).
- For qubits, Werner states are noisy singlets for two-qubits

$$\varrho_{\text{Werner}}(p) = (1-p)|\Psi_{\text{singlet}}\rangle\langle\Psi_{\text{singlet}}| + p\frac{\mathbb{1}}{4}.$$

- For two qudits, their density matrix is defined as

$$\alpha\mathbb{1} + \beta F$$

  where F is the flip operator. For systems larger than qubits, we do not have a pure Werner state.
- Twirling

$$\tau(\varrho) = \int M(dU) U \otimes U \varrho (U \otimes U)^{\dagger}.$$

  Used to transform states into a normal form, for example, before distilling entanglement. Twirling leaves Werner states unchanged. It transforms all quantum states to Werner states.

Werner states II

- Multipartite Werner states are defined as states that are invariant under $U^{\otimes N}$.
- For $d = 3$, there are three-qudit Werner states that are pure. Such a state is the fermionic singlet

$$\Psi_{\mathrm{fs}} = \frac{1}{\sqrt{6}}(|123\rangle - |132\rangle + -...).$$

- There are entangled Werner states that do not violate any Bell inequality. (See Bell inequalities later.)

Outline

Isotropic states

- Isotropic states are invariant under any transformation of the type $U \otimes U^*$.
- Isotropic states are defined as the maximally entangled state

$$|\Psi_{\mathrm{me}}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |kk\rangle.$$

  mixed with white noise.
- The maximally entangled state above is the pure isotropic state for any dimension.

Outline

1. **Interesting quantum states**
   - Motivation
   - A. Single particle states
   - B. Bipartite singlet state
   - C. Werner states
   - D. Isotropic states
   - E. Schrödinger cat states
   - F. Greenberger-Horne-Zeilinger (GHZ) state
   - G. W state
   - H. Symmetric Dicke states

Schrödinger cat states

- Think on the usual the Schrödinger's cat experiment. The cat is in a superposition of being dead and alive.
- Questions about the linearity of quantum mechanics, etc. Can superpositions of macroscopically different objects exist?
- Funny thought-experiment: Wigner's friend.

Outline

1. **Interesting quantum states**
   - Motivation
   - A. Single particle states
   - B. Bipartite singlet state
   - C. Werner states
   - D. Isotropic states
   - E. Schrödinger cat states
   - F. Greenberger-Horne-Zeilinger (GHZ) state
   - G. W state
   - H. Symmetric Dicke states

Greenberger-Horne-Zeilinger (GHZ) state

- A possible generalization of the maximally entangled state to N qubits is the GHZ state defined as

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|00...00\rangle + |11..11\rangle\right)$$

- The entanglement of the GHZ state is very fragile. Measuring one qubit destroys it, the state becomes separable:

$$\mathrm{Tr}_1(|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N|) = \frac{1}{2}(|0\rangle^{\otimes(N-1)} + |1\rangle^{\otimes(N-1)}).$$

- Realized experimentally with trapped ions, photons $\sim 10 - 15$ qubits.

Greenberger-Horne-Zeilinger (GHZ) state II

- Interestingly, the GHZ state is an eigenstate of operators that are the products of single-qubit operators. For example, it is the eigenstate of

$$\sigma_x \otimes \sigma_x \otimes ... \otimes \sigma_x = \sigma_x^{\otimes N}$$

with an eigenvalue $+1$. What does this mean? If they flip all the qubits, we get back the original states.

- The GHZ state is also the eigenstate of the operators of the type

$$\sigma_z^{(m)} \sigma_z^{(n)},$$

for all $m \neq n$ with an eigenvalue $+1$.

Greenberger-Horne-Zeilinger (GHZ) state III

- If a state is eigenstate of operators O_1 and O_2, then it is also an eigenstate of $O_1 O_2$. Because of that the state is an eigenstate of the products of such operators.
- Note that all these operators commute with each other.
- These operators form a *group* that is called *stabilizer.* The group has N generators and 2^N elements.
- (Definition of a discrete group: If A and B are in the group, so is their product AB.)
- Stabilizer theory is important for quantum error correction.

Greenberger-Horne-Zeilinger (GHZ) state IV

- Just an example: for three-qubit GHZ states, we have the following 8 operators

$$
\begin{aligned}
\sigma_x \otimes \sigma_x \otimes \sigma_x, \\
\sigma_z \otimes \sigma_z \otimes \mathbb{1}, \\
\mathbb{1} \otimes \sigma_z \otimes \sigma_z, \\
\sigma_z \otimes \mathbb{1} \otimes \sigma_z, \\
-\sigma_y \otimes \sigma_y \otimes \sigma_x, \\
-\sigma_x \otimes \sigma_y \otimes \sigma_y, \\
-\sigma_y \otimes \sigma_x \otimes \sigma_y, \\
\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}.
\end{aligned}
$$

Outline

W state

- It is defined as

$$\frac{1}{\sqrt{N}}\left(|100...\rangle + |010...\rangle + |001...\rangle + ... + |000..01\rangle\right).$$

  Interestingly, it maximizes the two-body concurrence among symmetric states.

- I we loose a single particle, we obtain

$$\mathrm{Tr}_1(|\mathrm{W}_N\rangle\langle\mathrm{W}_N|) = \frac{1}{N}|0\rangle\langle 0|^{\otimes(N-1)} + \frac{N-1}{N}|\mathrm{W}_{N-1}\rangle\langle\mathrm{W}_{N-1}|.$$

  The entanglement of the W state is robust. Loosing one qubit does not destroy the entanglement.

- Realized with trapped ions, photons, cold atoms, up to thousands of particles.

Outline

Symmetric Dicke states of qubits

- Dicke states are simultaneous eigenstate of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.
- Symmetric Dicke states of qubits have a maximal $\vec{J}^2$. They are the equal superpositions of the permutations of a series of 0's and 1's

$$|D_N^{(m)}\rangle \propto \sum_k \mathcal{P}_k(|0\rangle^{\otimes(N-m)} \otimes |1\rangle^{\otimes(m)}).$$

- If half of the particles are 1, half of them are zero, we have $m = N/2$. In this case, $\langle J_z \rangle = 0$.
- For example, for $N = 4$ it is

$$|D_4^{(2)}\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + \text{all permutations}).$$

- Relation to W states: $|D_N^{(1)}\rangle = |W_N\rangle$.
- Realized with trapped ions, photons, cold gases, up to thousands of particles.