Activating hidden metrological usefulness

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Outline

1. **Motivation**
   - What are entangled states useful for?

2. **Metrological gain and the optimal local Hamiltonian**
   - Metrological usefulness of a quantum state.
   - Activation of metrological usefulness
   - Optimal local Hamiltonian
   - Bipartite pure entangled states

What are entangled states useful for?

- Entanglement is needed for beating the shot-noise limit in quantum metrology.

- However, not all entangled states are more useful than separable states.

- Intriguing questions:

  - Can a quantum state become useful metrologically, if an ancilla or a second copy is added?

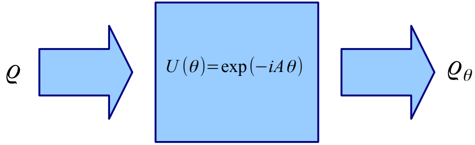
  - How to find the local Hamiltonian, for which a quantum state is the most useful compared to separable states?

Outline

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{mF_Q[\varrho, A]},$$



where where m is the number of independent repetitions and $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states of N qubits

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

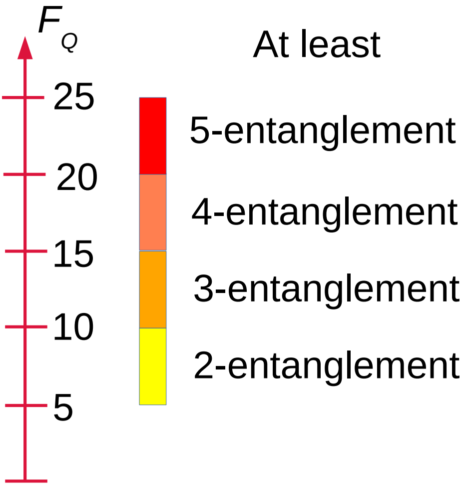
- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

The quantum Fisher information vs. entanglement

5 spin-1/2 particles



Metrological usefulness

- Metrological gain for a given Hamiltonian

$$g_{\mathcal{H}}(\varrho) = \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})},$$

where $\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})$ is the maximum of the QFI for separable states.

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\mathrm{local}\mathcal{H}} g_{\mathcal{H}}(\varrho) = \max_{\mathrm{local}\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H})}.$$

- A state ϱ is useful if $g(\varrho) > 1$.

- The metrological gain is convex in the state.
[G. Toth, T. Vertesi, P. Hordecki, R. Horodecki, PRL 2020.]

- We would like to detmine g.

Maximally entangled state

- Difficult to obtain $g(\varrho)$ and the optimal local Hamiltonian for any ϱ.
- As a first step, we consider the $d \times d$ maximally entangled state

$$|\Psi^{(\mathrm{me})}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle|k\rangle.$$

- The optimal Hamiltonian is

$$\mathcal{H}^{(\mathrm{me})} = D \otimes \mathbb{1} + \mathbb{1} \otimes D,$$

where

$$D = \mathrm{diag}(+1, -1, +1, -1, ...).$$

Maximally entangled state II

The 3×3 noisy quantum state

$$\varrho_{AB}^{(p)} = (1-p)|\Psi^{(\mathrm{me})}\rangle\langle\Psi^{(\mathrm{me})}| + p\mathbb{1}/d^2,$$

is useful if

$$p < \frac{25-\sqrt{177}}{32} \approx 0.3655,$$

while for larger p's it is not useful.

- Note that it is entangled if

$$p < \frac{2}{3}.$$

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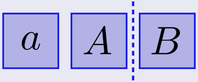
Activation by an ancilla qubit

If a pure ancilla qubit is added

$$\varrho^{(\text{anc})} = |0\rangle\langle 0|_a \otimes \varrho_{AB}^{(p)}.$$



then the state is useful if

$$p < 0.3752.$$

(For a single copy, the limit was $p < 0.3655$.)

- The Hamiltonian is

$$\mathcal{H}^{(\text{anc})} = 1.2 C_{aA} \otimes \mathbb{1}_B + \mathbb{1}_{aA} \otimes D_B,$$

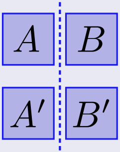
where

$$C_{aA} = \frac{9}{20} (2\sigma_x + \sigma_z)_a \otimes |0\rangle\langle 0|_A + \mathbb{1}_a \otimes (|2\rangle\langle 2|_A - |1\rangle\langle 1|_A).$$

Activation by a second copy

If a second copy is added

$$\varrho^{(\mathrm{tc})} = \varrho^{(p)}_{AB} \otimes \varrho^{(p)}_{A'B'}.$$



then the state is useful if

$$p < 0.4164.$$

(For a single copy, the limit was $p < 0.3655$.)

- The Hamiltonian is

$$\mathcal{H}^{(\mathrm{tc})} = D_A \otimes D_{A'} \otimes \mathbb{1}_{BB'} + \mathbb{1}_{AA'} \otimes D_B \otimes D_{B'}.$$

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Method for maximizing g

- Metrological gain optimized over all local Hamiltonians

$$g(\varrho) = \max_{\text{local}\mathcal{H}} \frac{\mathcal{F}_Q[\varrho, \mathcal{H}]}{\mathcal{F}_Q^{(\text{sep})}(\mathcal{H})} \begin{array}{l} \leftarrow \text{metrological performance of } \varrho \\ \leftarrow \text{best metrological performance of} \\ \quad \text{separable states} \end{array}$$

- It is a fundamental quantity metrology!
- Difficult to compute, since $\mathcal{H}$ is in both the numerator and the denominator!
- We reduce the problem to maximize $\mathcal{F}_Q$ over a set of local Hamiltonians.

Method for finding the optimal local Hamiltonian I

- Direct maximization of $\mathcal{F}_Q[\varrho,\mathcal{H}]$ over $\mathcal{H}$ is difficult: it is convex in $\mathcal{H}$.

- Let us consider the error propagation formula

$$(\Delta\theta)^2{}_M = \frac{(\Delta M)^2}{\langle i[M,\mathcal{H}]\rangle^2},$$

which provides a bound on the quantum Fisher information

$$\mathcal{F}_Q[\varrho,\mathcal{H}] \geq 1/(\Delta\theta)^2{}_M.$$

[M. Hotta and M. Ozawa, Phys. Rev. A 2004; B. M. Escher, arXiv:1212.2533; F. Fröwis, R. Schmied, and N. Gisin, Phys. Rev. A 2015. For a summary, see, e.g., the Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020.]

Method for finding the optimal Hamiltonian II

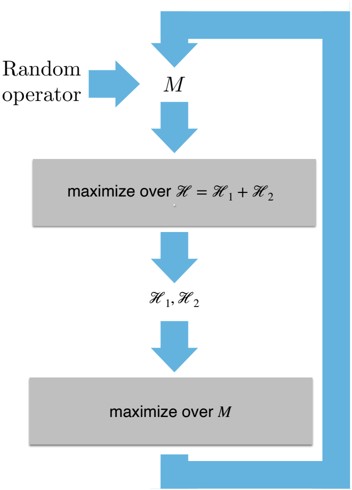
The maximum over local Hamiltonians can be obtained as

$$\max_{\text{local } \mathcal{H}} \mathcal{F}_Q[\varrho, \mathcal{H}] = \max_{\text{local } \mathcal{H}} \max_{M} \frac{\langle i[M, \mathcal{H}] \rangle_\varrho^2}{(\Delta M)^2}.$$

Similar idea for optimizing over the state, rather than over $\mathcal{H}$:

[K. Macieszczak, arXiv:1312.1356; K. Macieszczak, M. Fraas, and R. Demkowicz-Dobrzański, New J. Phys. 16, 113002 (2014); Tóth and Vértesi, Phys. Rev. Lett. (2018).]

See-saw algorithm



The precision cannot get worse with the iteration!

Note that $\mathcal{H}_1, \mathcal{H}_2$ fulfill

$$c_n \mathbb{1} \pm \mathcal{H}_n \geq 0.$$

Numerical results

- We remember that the 3×3 isotropic state is useful if

$$p < \frac{25 - \sqrt{177}}{32} \approx 0.3655.$$

- Then, we have the following results for activation.

	Analytic example	Numerics
Ancilla	0.3752	0.3941
Second copy	0.4164	0.4170

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Single copy of pure states

All entangled bipartite pure states are metrologically useful.

- *Proof.*—For the two-qubit case, see
  P. Hyllus, O. Gühne, and A. Smerzi, Phys. Rev. A 82, 012337 (2010).

- General case, pure state with a Schmidt decomposition

$$|\Psi\rangle = \sum_{k=1}^{s} \sigma_k |k\rangle_A |k\rangle_B,$$

  where s is the Schmidt number, and the real positive σ_k Schmidt coefficients are in a descending order.

- We define

$$\mathcal{H}_A = \sum_{n=1,3,5,\ldots,\tilde{s}-1} |+\rangle\langle+|_{A,n,n+1} - |-\rangle\langle-|_{A,n,n+1},$$

  where $\tilde{s}$ is the largest even number for which $\tilde{s} \leq s$, and

$$|\pm\rangle_{A,n,n+1} = (|n\rangle_A \pm |n+1\rangle_A)/\sqrt{2}.$$

Single copy of pure states II

- We define $\mathcal{H}_B$ in a similar manner.

- We also define the collective Hamiltonian

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}_B.$$

  Then, we have $\langle \mathcal{H}_{AB} \rangle_\Psi = 0$.

- Direct calculation yields

$$\mathcal{F}_Q[|\Psi\rangle, \mathcal{H}_{AB}] = 4(\Delta \mathcal{H}_{AB})^2{}_\Psi = 8 \sum_{n=1,3,5,\ldots,\tilde{s}-1} (\sigma_n + \sigma_{n+1})^2,$$

  which is larger than the separable bound, $\mathcal{F}_Q^{(\mathrm{sep})} = 8$, whenever the Schmidt rank is larger than 1. □

Infinite number of copies

In the infinite copy limit, all bipartite pure entangled states are maximally useful.

[Supplemental Material of Tóth, Vértesi, Horodecki, Horodecki, PRL 2020.]

Summary

- Some entangled quantum states that are not useful metrologically, can still be made useful, if an ancilla or an additional copy is added.
- We have shown a general method to get the optimal local Hamiltonian for a quantum state.
- All bipartite pure entangled states are useful metrologically. If infinite number of copies are given, they are maximally useful.

See:

Géza Tóth, Tamás Vértesi, Paweł Horodecki, Ryszard Horodecki,

Activating hidden metrological usefulness,

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THANK YOU FOR YOUR ATTENTION!

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