Introduction to entanglement theory & Detection of multipartite entanglement close to symmetric Dicke states

G. Tóth[1,2,3]

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Murmansk
Norwegian Sea
Arctic Circle
Reykjavík
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White Sea
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Athens
BALEARIC ISLANDS
Sevilla
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Cagliari
Ionian Sea
Gibraltar (U.K.)
Málaga
Palermo
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

Why multipartite entanglement is important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.

Notation

- Qubit: $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$,
- Density matrix: $\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$,
- Expectation value: $\langle A\rangle = \mathrm{Tr}(\varrho \mathrm{A})$.
- Variance: $(\Delta A)^2 = \mathrm{Tr}(\varrho \mathrm{A}^2) - \mathrm{Tr}(\varrho \mathrm{A})^2$.

Defintion of Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

R.F. Werner, Phys. Rev. A 1989

Questions for multipartite entanglement

For many particles, it is not sufficient to say “entangled”/”not entangled”.

We have to have more than these two levels.

k-producibility/k-entanglement

A pure state is *k-producible* if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and GT, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

Genuine multipartite entanglement

- N-particle entanglement $\equiv$ genuine multipartite entanglement.

Usefulness of entanglement

- Entangled states are useful for quantum cryptography, for quantum teleportation.
- Entangled states outperform separable ones in very general tasks in quamtum metrlogy.
- Note: Entanglement cannot be obtained from separable states with local operations and classical communications (LOCC). It is a resource.

Entanglement detection

- It is a very hard task to decide whether a quantum state is separable or not.
- Solved for small systems: 2×2, 2×3. (Peres-Horodecki condition, 1997.)
- In general, necessary conditions for separability exist. If they are violated, then we know that the state is entangled

Examples

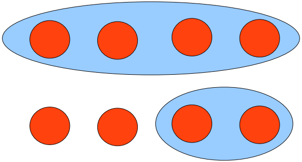
Examples

Two entangled states of four qubits:

$$|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

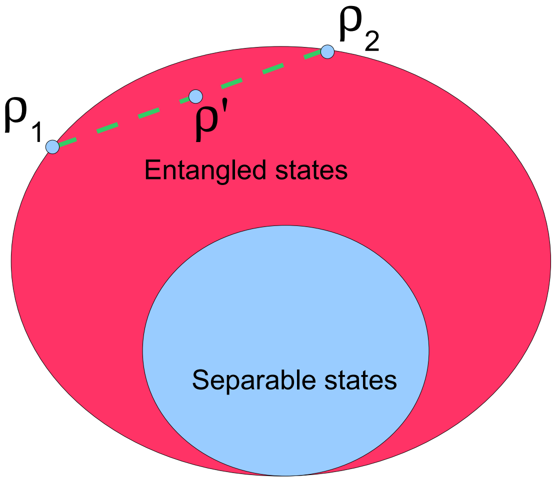
- The first state is genuine multipartite entangled, the second state is biseparable.



Outline

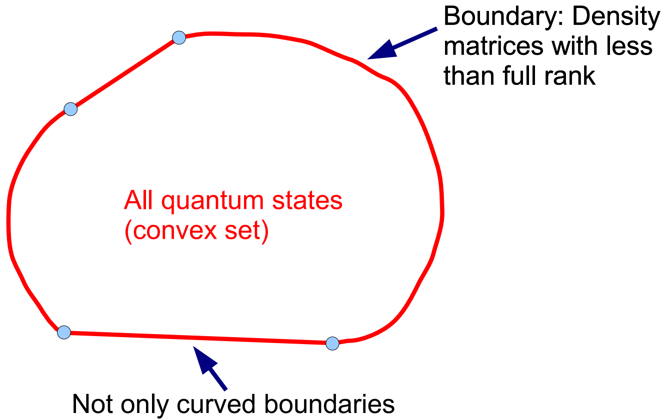
Theory of quantum entanglement

- Separable states form a convex set.



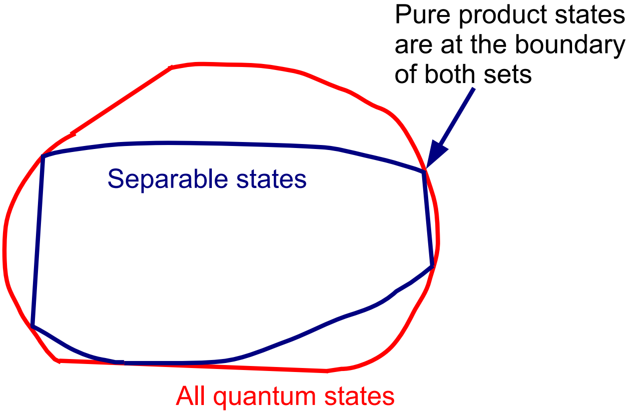
Theory of quantum entanglement II

- A more accurate picture:



Theory of quantum entanglement III

- Together with the set of separable states:



Outline

Witnesses based on correlations

Definition

An entanglement witness $\mathcal{W}$ is an operator that is positive on all separable (biseparable) states.

Thus, $\mathrm{Tr}(\mathcal{W}\varrho) < 0$ signals entanglement (genuine multipartite entanglement).
[Horodecki 1996; Terhal 2000; Lewenstein, Kraus, Cirac, Horodecki 2002]

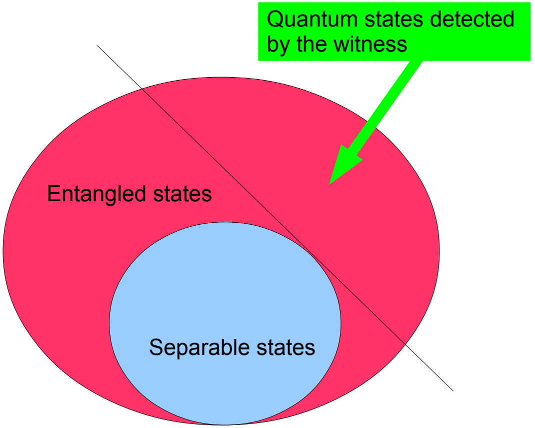
There are two main goals when searching for entanglement witnesses:

Optimization → Large robustness to noise

Optimization → Few measurements

Convex sets for the entanglement witnesses

- Entanglement witnesses in the convex set picture



Witnesses based on correlations

Example

Witness with Heisenberg interaction

$$\mathcal{W}_{xyz} = \mathbb{1} \otimes \mathbb{1} + \sigma_x^{(1)} \otimes \sigma_x^{(2)} + \sigma_y^{(1)} \otimes \sigma_y^{(2)} + \sigma_z^{(1)} \otimes \sigma_z^{(2)}.$$

Proof. For product states of the form $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$, we have

$$\langle \sigma_x \otimes \sigma_x \rangle + \langle \sigma_y \otimes \sigma_y \rangle + \langle \sigma_z \otimes \sigma_z \rangle = \sum_{l=x,y,z} \langle \sigma_l \rangle_{\Psi_1} \langle \sigma_l \rangle_{\Psi_2} \geq -1.$$

Here, we used the Cauchy-Schwarz inequality. Due to convexity, the inequality is also true for separable states.

The minimum for quantum states is –3. Such a maximum is obtained for the state

$$\frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right).$$

Outline

Variance-based criteria

For a bipartite system, for both parties

$$(\Delta A_k)^2 + (\Delta B_k)^2 \geq L_k.$$

For product states of the form $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$, we have

$$(\Delta(A_1 + A_2))^2 = \langle (A_1 + A_2)^2 \rangle - \langle A_1 + A_2 \rangle^2 = (\Delta A_1)^2_{\Psi_1} + (\Delta A_2)^2_{\Psi_2}$$

since for product states

$$\langle A_1 A_2 \rangle - \langle A_1 \rangle \langle A_2 \rangle = 0.$$

Hence,

$$(\Delta(A_1 + A_2))^2 + (\Delta(B_1 + B_2))^2 \geq L_1 + L_2.$$

This is also true for separable states due to the convexity of separable states.

[See Gühne, Phys. Rev. Lett. (2004) for an exhaustive study.]

Variance-based criteria II

Example: we have

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}.$$

Hence,

$$(\Delta x)^2 + (\Delta p)^2 \geq 1.$$

Then, for two-mode separable states

$$(\Delta(x_1 + x_2))^2 + (\Delta(p_1 - p_2))^2 \geq 2.$$

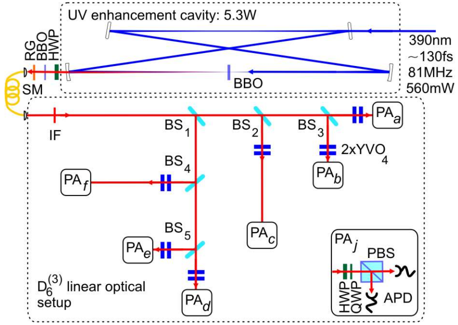
Any state violating this is entangled.
[Generalization: L.M. Duan, G. Giedke, J.I. Cirac, P. Zoller, Phys. Rev. Lett (2000); R. Simon, Phys. Rev. Lett (2000).]

Outline

Experiment with photons

- A photon can have a horizontal (H) and a vertical (V) polarization.
- H/V can take the role of 0 and 1.
- Problem: photons do not interact with each other.

Photons II

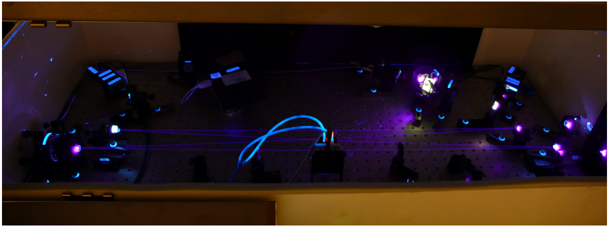


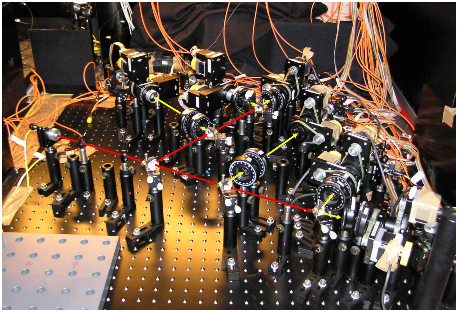
MPQ, Munich. Experiments with 6 photons.

[W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, Phys. Rev. Lett. 2009.]

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{20}} (|111000\rangle + |110100\rangle + ... + |000111\rangle).$$

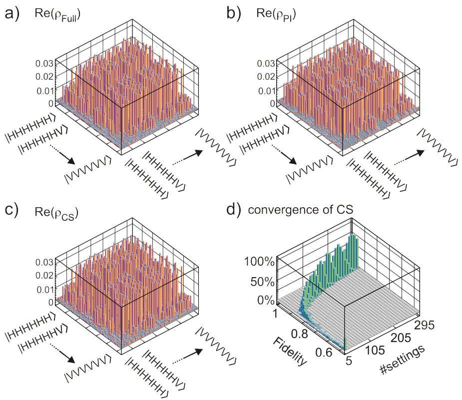
Photons III





Photons IV

6-qubit Quantum state tomography



[C. Schwemmer, G. Tóth, A. Niggebaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, Phys. Rev. Lett 113, 040503 (2014).]

Photons VI

- Entanglement witness for Detecting genuine multipartite entanglement close to Dicke states

$$\mathcal{W}_{\mathrm{D}} := \tfrac{3}{5}\mathbb{1} - |D_6^{(3)}\rangle\langle D_6^{(3)}|,$$

where

$$|D_6^{(3)}\rangle = \frac{1}{\sqrt{20}}\left(|111000\rangle + |110100\rangle + ... + |000111\rangle\right).$$

- If we have

$$\langle \mathcal{W}_{\mathrm{D}} \rangle < 0$$

then the state is genuine multipartite entangled.

[G. Tóth, JOSAB 2007.]

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

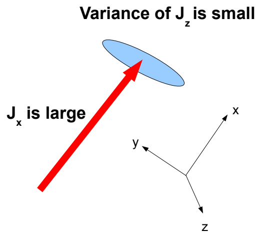
The standard spin-squeezing criterion

The spin squeezing parameter is defined as

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

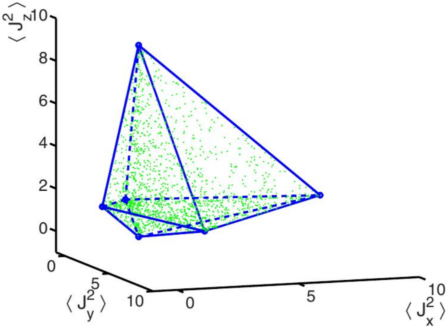
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{\mathrm{p2}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{\mathrm{p2}}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable ϱ_{p2}.
- Still, as we will see, we can even detect multipartite entanglement!

Outline

Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply "Dicke states" in the following) are defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel *et al.,* Phys. Rev. Lett. 2007; Prevedel *et al.*, Phys. Rev. Lett 2007; Wieczorek *et al.*, Phys. Rev. Lett. 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011]

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

[GT, JOSAB 2007.]

- ... are optimal for quantum metrology, similarly to GHZ states.

[Hyllus *et al.*, PRA 2012;Lücke *et al.*, Science 2011.]

[GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.]

- ... are macroscopically entangled, like GHZ states.

[Fröwis, Dür, PRL 2011]

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle - \tfrac{N}{2} \leq (N-1)(\Delta J_m)^2.$$

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \mathrm{max.},$$
$$\langle J_z^2 \rangle = 0.$$

Spin Squeezing Inequality for Dicke states II

Based on the above inequality, we define a new spin squeezing parameter

$$\xi_{\mathrm{os}}^2 = \frac{RHS}{LHS} = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2 \rangle - \frac{N}{2}}.$$

[Vitagliano, Apellaniz, Egusquiza, GT, PRA (2014)]

- For our Dicke state, the numerator is minimal, the denominator is maximal, $\xi_{\mathrm{os}}^2 = 0$.
- The original spin squeezing parameter would not detect the Dicke states, since

$$\xi_{\mathrm{s}}^2 = N\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2} = N\frac{(\Delta J_z)^2}{0} = \infty.$$

Fully polarized states

- Relation between the second moments and the expectation value

$$\langle J_x^2 \rangle = \langle J_x \rangle^2 + (\Delta J_x)^2 \geq \langle J_x \rangle^2.$$

- For states polarized in the x-direction and spin squeezed along the z-direction, for $N \gg 1$, we have

$$\langle J_x^2 \rangle \approx \langle J_x \rangle^2 \gg N.$$

Hence, for fully polarized states

$$\xi_{\mathrm{os}}^2 = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2 \rangle - \frac{N}{2}} \approx \xi_{\mathrm{s}}^2 = N\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

Outline

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{n=1}^{M} |\psi^{(n)}\rangle,$$

where $|\psi^{(n)}\rangle$ is the state of at most k qubits.

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

[Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: C. Gross *et al.*, Nature 464, 1165 (2010).]

Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.
- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, arxiv (2014).

Multipartite entanglement - Our condition

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x\rangle^2 + \langle J_y\rangle^2}}{J_{\max}}\right).$$

- Combine it with

$$\langle J_x^2 + J_y^2\rangle \leqslant J_{\max}(\tfrac{k}{2}+1) + \langle J_x\rangle^2 + \langle J_y\rangle^2,$$

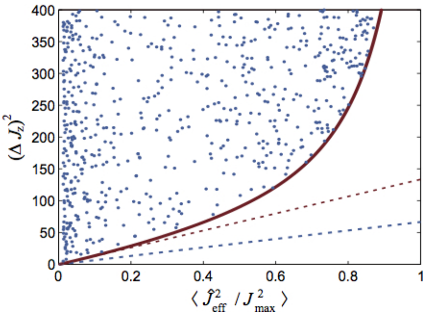
which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x^2 + J_y^2\rangle - J_{\max}(\frac{k}{2}+1)}}{J_{\max}}\right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

Concrete example



- $N = 8000$ particles, and $J_{\mathrm{eff}} = J_x^2 + J_y^2$.
- Red curve: boundary for 28-particle entanglement.
- Blue dashed line: linear condition given in [L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).]
- Red dashed line: tangent of our curve.

Outline

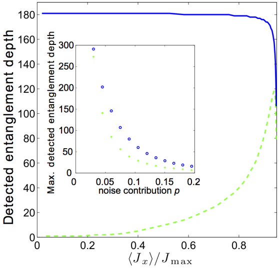
Our condition is stronger

- Consider spin squeezed states as ground states of

$$H(\Lambda) = J_z^2 - \Lambda J_x.$$

For $\Lambda = \infty$, the ground state is fully polarized. For $\Lambda = 0$, it is the symmetric Dicke state.

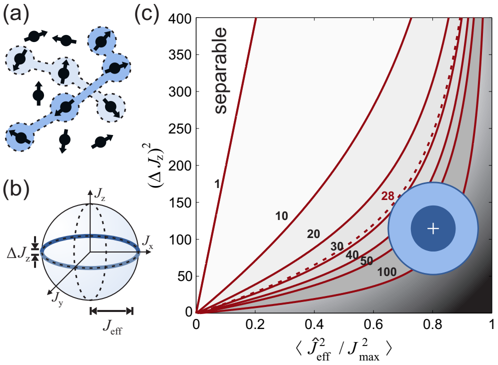
- Our condition vs. original condition for N=4000 and p=0.05



[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014).]

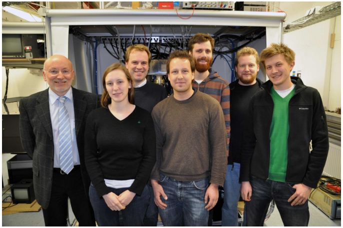
Experimental results

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014).]

Project participants



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G. Vitagliano

I. Apellaniz

L. Santos

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I.L. Egusquiza

Bilbao

Summary

- Detection of multipartite entanglement close to Dicke states, by measuring collective quantities only.
- The condition detects all entangled states that can be detected based on the measured quantities (i.e., it is optimal).

Vitagliano, Apellaniz, Egusquiza, GT, PRA (2014).

Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt,
PRL 112, 155304 (2014)
(synopsis at physics.aps.org, Revista Española de Física).

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