Quantum Fisher information and entanglement

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F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Hammerfest
Tromsø
Murmansk
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
Kiruna
White Sea
Arkhangelsk
NORWAY
Luleå
Oulu
Umeå
FINLAND
Trondheim
SWEDEN
Gulf of Bothnia
Tampere
Lake Ladoga
Tórshavn
Faroe Islands (DENMARK)
Gävle
Turku
Helsinki
Gulf of Finland
St. Petersburg
SHETLAND ISLANDS
Bergen
Oslo
ALAND ISLANDS
Stockholm
Tallinn
ESTONIA
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Rockall (U.K.)
ORKNEY ISLANDS
HEBRIDES
Stavanger
North Atlantic Ocean
Aberdeen
Göteborg
Gotland
Riga
LATVIA
Glasgow
Edinburgh
North Sea
DENMARK
Copenhagen
Malmö
Baltic Sea
Öland
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Smolensk
Belfast
UNITED KINGDOM
Kaliningrad
RUSSIA
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Mahilyow
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Irish Sea
Isle of Man (U.K.)
Liverpool
Leeds
Manchester
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Gdańsk
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MONACO
SAN MARINO
BULGARIA
Sofia
Porto
Andorra la Vella
ANDORRA
Marseille
Ligurian Sea
Florence
Adriatic Sea
Istanbul
Zaragoza
Corsica
ITALY
Skopje
F.Y.R.O.M.
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Madrid
Barcelona
VATICAN CITY
ALB.
Thessaloniki
Lisbon
Balearic Sea
Naples
SPAIN
Valencia
Sardinia
Tyrrhenian Sea
GREECE
Aegean Sea
BALEARIC ISLANDS
Athens
Sevilla
Ionian Sea
Mediterranean Sea
Cagliari
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Algiers
Tunis
Melilla (SPAIN)
Oran
Crete
Rabat
Casablanca
TUNISIA
Valletta
MALTA
MOROCCO
ALGERIA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

Motivation

- The quantum Fisher information (QFI) plays a central role in metrology.
- In linear interferometers, the QFI is directly related to multipartite entanglement.
- Thus, one can detect entanglement with precision measurements.

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

k-producibility/k-entanglement

A pure state is k-producible if it can be written as

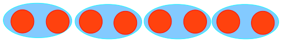
$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

where $|\Phi_l\rangle$ are states of at most k qubits.

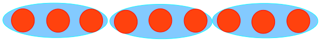
A mixed state is k-producible, if it is a mixture of k-producible pure states.

e.g., Gühne, GT, NJP 2005.

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.



two-producible

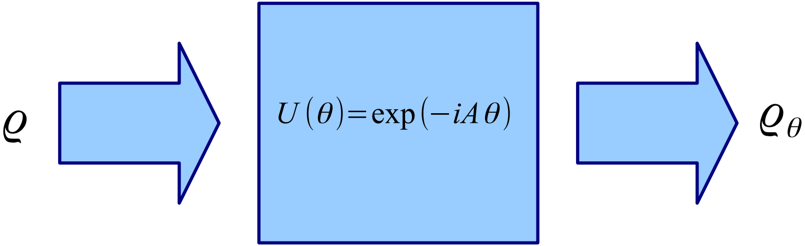


three-producible

Outline

Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2 := F_Q(\varrho; i[\varrho, A]),$$

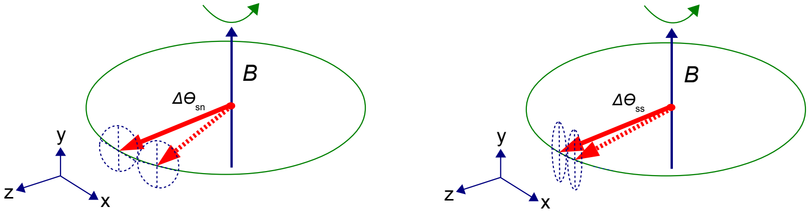
where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Special case $A = J_l$

- The operator A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer



Outline

The quantum Fisher information vs. entanglement

- For pure product states of N spin-$\frac{1}{2}$ particles

$$|\Psi\rangle = |\psi^{(1)}\rangle \otimes |\psi^{(2)}\rangle \otimes |\psi^{(3)}\rangle \otimes ... \otimes |\psi^{(N)}\rangle,$$

we have

$$F_Q[\varrho, J_z] = 4(\Delta J_z)^2 = 4\sum_{n=1}^{N}(\Delta j_z^{(n)})^2 \leq N.$$

- For separable states (mixtures of pure product states) we have

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z,$$

since $F_Q[\varrho, A]$ is convex in ϱ.

Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009);
Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010).

The quantum Fisher information vs. multipartite entanglement

- For N-qubit k-producible states states, the quantum Fisher information is bounded from above by

$$F_Q[\varrho, J_l] \leq nk^2 + (N - nk)^2.$$

where n is the integer part of $\frac{N}{k}$.

- If k is divisor of N then

$$F_Q[\varrho, J_l] \leq kN.$$

P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012).

The quantum Fisher information vs. macroscopic superpositions

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2,$$

F. Fröwis and W. Dür, New J. Phys. 14 093039 (2012).

Summary of various types of limits

Bounds for the QFI

- Shot-noise limit: $F_Q[\varrho, J_l] \leq N$,
- Heisenberg limit: $F_Q[\varrho, J_l] \leq N^2$.

Bounds for the precision

- Shot-noise limit: $(\Delta\theta)^2 \geq \frac{1}{N}$,
- Heisenberg limit: $(\Delta\theta)^2 \geq \frac{1}{N^2}$.

Scaling of the precision in a noisy environment

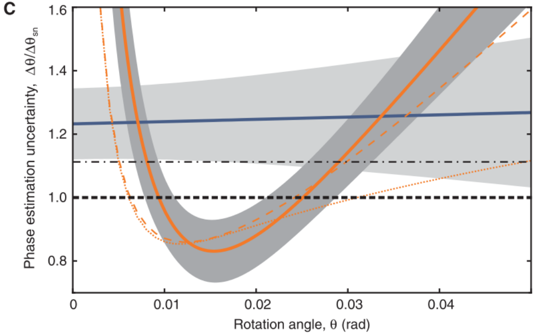
- Is the scaling $F_Q[\varrho, J_l] \propto N^2$ possible? Too good to be true?
- One feels that this is probably not possible.
- Due to uncorrelated local noise the scaling returns to the shot-noise scaling

$$F_Q[\varrho, J_l] \leq \text{const.} \times N$$

R. Demkowicz-Dobrzański J. Kołodyński, M. Guţă, Nat. Commun. 3, 1063 (2012); B. Escher, R. de Matos Filho, L. Davidovich, Nat. Phys. 7, 406 (2011).

Entanglement detection with precision measurement

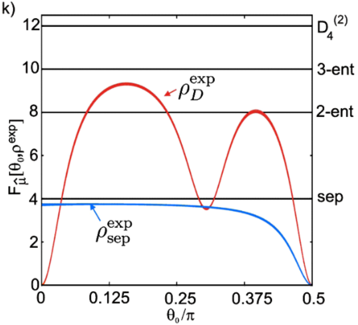
- Entanglement detection in cold gases.



B. Lücke *et al.*, Science, Science 334, 773 (2011).

Entanglement detection with precision measurement II

- Entanglement in a photonic experiment.



Krischek *et al.*, Phys. Rev. Lett. 107, 080504 (2011)

Outline

Families of the Quantum Fisher information

Definition

The quantum Fisher information is defined as

$$\hat{F}^f(\varrho; A) = \mathrm{Tr}(A \mathbb{J}_f^{-1}(\varrho) A),$$

where

$$\mathbb{J}_\varrho^f(A) = f(\mathbb{L}_\varrho {\mathbb{R}_\varrho}^{-1})\mathbb{R}_\varrho,$$

and

$$\mathbb{L}_\varrho(A) = \varrho A, \quad \mathbb{R}_\varrho(A) = A\varrho.$$

- $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a standard operator monotone function, which has the properties $xf(x^{-1}) = f(x)$ and $f(1) = 1$.
- Mean based on f
$$m_f(a, b) = af\left(\frac{b}{a}\right).$$

Families of the Quantum Fisher information II

- Form with density matrix eigenvalues and eigenvectors

$$\hat{F}_Q^f(\varrho; A) = \sum_{i,j} \frac{1}{m_f(\lambda_i, \lambda_j)} |\langle i|A|j\rangle|^2.$$

The ususal QFI, $\hat{F}_Q(\varrho; A) \neq F_Q[\varrho, A]$

- For the arithmetic mean $m_f(a,b) = \frac{a+b}{2}$,

$$\hat{F}_Q(\varrho; A) = \sum_{i,j} \frac{2}{\lambda_i + \lambda_j} |A_{ij}|^2.$$

- $\hat{F}(\varrho; A)$ is the smallest among the various types of the generalzied quantum Fisher information. [Normalization f(1)=1.]
- The quantum Fisher information is defined for the linear dynamics

$$\varrho_{\text{output}}(t) = \varrho + At$$

- Cramer Rao bound: $(\Delta t)^2 \geq 1/\hat{F}(\varrho; A)$.

D. Petz, J. Phys. A: Math. Gen. **35**, 929 (2002);

P. Gibilisco, F. Hiai, and D. Petz, IEEE Trans. Inform. Theory **55**, 439 (2009).

Outline

Families of the variances

Definition

Generalized variance

$$\hat{\mathrm{var}}_{\varrho}^{f}(A) = \langle A, \mathbb{J}_{\varrho}^{f}(A)\rangle - (\mathrm{Tr}\,\varrho A)^2,$$

where f is the matrix monotone function mentioned before.

- For the arithmetic mean $m_f(a,b) = \frac{a+b}{2}$, we get the usual variance

$$\hat{\mathrm{var}}_{\varrho}^{f}(A) = \langle A^2\rangle_{\varrho} - \langle A\rangle_{\varrho}^{2}.$$

- It is the largest among the generalized variances.

Families of the variances II

Form given with the density matrix eigenvalues and eigenvectors

$$\mathrm{var}_{\varrho}^{f}(A) = \sum_{i,j} m_f(\lambda_i, \lambda_j)|A_{ij}|^2 - \left|\sum \lambda_i A_{ii}\right|^2.$$

- The usual variance is the largest. (The arithmetic mean is the largest mean.)
- For pure states, they give

$$2m_f(1,0) \times (\Delta A)^2.$$

Outline

Family of the Quantum Fisher information for unitary dynamics

1. In physics, we are typically interested in a unitary dynamics

$$\varrho_{\rm output}(t) = \exp(-iA\theta)\varrho\exp(+iA\theta),$$

2. Then,

$$\hat{F}_Q^f[\varrho, A] = \hat{F}^f(\varrho; i[\varrho, A]).$$

3. We propose the normalization

$$F_Q^f[\varrho, A] = 2m_f(1,0)\hat{F}_Q^f[\varrho, A]$$

4. For a pure state $|\Psi\rangle$ we have

$$F_Q^f[\varrho, A] = 4(\Delta A)^2{}_\Psi.$$

Family of the Quantum Fisher information for unitary dynamics II

The generalized QFIs given with the density matrix eigenvalues

$$F_Q^f[\varrho, A] = 2\sum_{i,j} \frac{m_f(1,0)}{m_f(\lambda_i, \lambda_j)} (\lambda_i - \lambda_j)^2 |A_{ij}|^2.$$

- For pure states, equals $4(\Delta A)^2$.
- The usual QFI is the largest. (The arithmetic mean is the largest mean.)
- Exactly the opposite of what we had with the original normalization!

The ususal QFI, unitary dynamics

- For the arithmetic mean $m_f(a,b) = \frac{a+b}{2}$,

$$F_Q[\varrho, A] = \sum_{i,j} \frac{2}{\lambda_i + \lambda_j}(\lambda_i - \lambda_j)^2 |A_{ij}|^2.$$

  "THE" quantum Fisher Information.

- The quantum Fisher information is defined for the linear dynamics

$$\varrho_{\text{output}}(t) = e^{-iA\theta} \varrho e^{+iA\theta}.$$

- Cramer Rao bound: $(\Delta\theta)^2 \geq 1/F[\varrho, A]$.

Family of the Quantum Fisher information for unitary dynamics IV

Definition

Generalized quantum Fisher information $F_Q[\varrho, A]$

1. For pure states, we have

$$F_Q[|\Psi\rangle\langle\Psi|, A] = 4(\Delta A_\Psi)^2.$$

   The factor 4 appears to keep the consistency with the existing literature.

2. For mixed states, $F_Q[\varrho, A]$ is convex in the state.

Family of variances

1. We propose the normalization

$$\mathrm{var}_\varrho^f(A) = \frac{\hat{\mathrm{var}}_\varrho^f(A)}{2m_f(1,0)}.$$

2. For a pure state $|\Psi\rangle$ we have

$$\mathrm{var}_\Psi^f(A) = (\Delta A)^2{}_\Psi.$$

Family of variances II

Form with density matrix eigenvalues

$$\mathrm{var}_{\varrho}^{f}(A) \;=\; \frac{1}{2}\sum_{i,j}\frac{m_f(\lambda_i,\lambda_j)}{m_f(1,0)}|A_{ij}|^2 - \Big|\sum \lambda_i A_{ii}\Big|^2 .$$

- For pure states, equals $(\Delta A)^2$.
- The usual variance is the smallest.
- Exactly the opposite of what we had with the original normalization!

Family of variances III

Definition

The generalized variance $\mathrm{var}_\varrho(A)$ is defined by the following two requirements.

1. For pure states, the generalized variance equals the usual variance
$$\mathrm{var}_\Psi(A) = (\Delta A)^2{}_\Psi.$$
2. For mixed states, $\mathrm{var}_\varrho(A)$ is concave in the state.

Families II

QFI:

original, linear dynamics, by D. Petz	unitary dynamics	unitary dynamics, our normalization
$\hat{F}^f(\varrho; A)$	$\hat{F}^f[\varrho, A] = \hat{F}^f(\varrho; i[\varrho, A])$	$F_Q^f[\varrho, A]$ $= 2m_f(1,0)\hat{F}_Q^f[\varrho, A]$
usual QFI is smallest	-	usual QFI $F_Q[\varrho, A]$ is largest

variance:

original, by D. Petz	our normalization
$\hat{\mathrm{var}}_\varrho^f(A)$	$\mathrm{var}_\varrho^f(A) = \frac{\hat{\mathrm{var}}_\varrho^f(A)}{2m_f(1,0)}$
usual variance is largest	usual variance is smallest

Outline

The QFI as a convex roof

- We have a family of generealized QFI's, which are convex in the state and all equal to four times the variance for pure states.
- There is a smallest of such functions defined by the convex roof.

The usual QFI equals

$$F_Q[\varrho, A] = 4 \inf_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k},$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013).

The variance as a concave roof

- We have a family of generealized variances, which are concave in the state and all equal to the variance for pure states.
- There is a largest of such functions defined by the concave roof.

The usual variance is

$$(\Delta A)^2{}_{\varrho} = \sup_{\{p_k, \Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k},$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

Summary of statements

- Decompose ϱ as
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$
- Then,
$$\frac{1}{4} F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2_{\Psi_k} \leq (\Delta A)^2_{\varrho}$$
holds.
- Both inequalities can be saturated by some decompostion..

Variance

- For 2×2 covariance matrices there is always $\{p_k.\Psi_k\}$ such that

$$C_\varrho = \sup_{\{p_k,\Psi_k\}} \sum_k p_k C_{\Psi_k},$$

[Z. Léka and D. Petz, Prob. and Math. Stat. 33, 191 (2013)]

- For 3×3 covariance matrices, this is not always possible. Necessary and sufficient conditions for an arbitrary dimension. [D. Petz and D. Virosztek, Acta Sci. Math. (Szeged) 80, 681 (2014)]

Why convex and concave roofs are interesting?

- Convex and concave roofs appear in entanglement theory (E.g., Entanglement of Formation).
- They do not often appear in other fields.
- Expression with convex and concave roofs typically cannot be computed with a single formula.
- The variance and the QFI are defined via roofs, but they can easily be calculated.

Outline

Witnessing the quantum Fisher information based on few measurements

- The bound based on $w = \mathrm{Tr}(\varrho W)$ is given as

$$F_Q[\varrho, J_z] \geq \sup_r \left[rw - \hat{\mathcal{F}}_{\mathrm{Q}}(rW) \right].$$

- The Legendre transform is

$$\hat{\mathcal{F}}_{\mathrm{Q}}(W) = \sup_{\varrho} (\langle W \rangle_{\varrho} - F_Q[\varrho, J_z]).$$

Optimization over ϱ: complicated.

- Due to the properties of F_Q mentioned before, it can be simplified

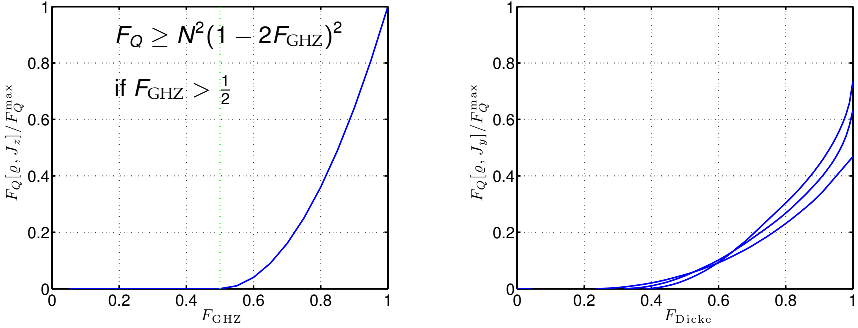
$$\hat{\mathcal{F}}_{\mathrm{Q}}(W) = \sup_{\mu} \left\{ \lambda_{\max} \left[W - 4(J_z - \mu)^2 \right] \right\}.$$

Optimziation over a single parameter!

I. Apellaniz, M. Kleinmann, O. Gühne, and G. Tóth, Phys. Rev. A 95, 032330 (2017), Editors' Suggestion.

Example: bound based on fidelity

- Let us bound the quantum Fisher information based on some measurements.



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

Apellaniz *et al.*, Phys. Rev. A 2017

Outline

QFI as a convex roof used for numerics

- The optimization over convex decompostions for $F_Q[\varrho, A]$ can be rerwrittem as

$$F_Q[\varrho, A] = 4\left(\langle A^2\rangle_\varrho - \sup_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k \langle A\rangle_{\Psi_k}^2\right),$$

  where $\{p_k, |\Psi_k\rangle\}$ refers to a decomposition of ϱ.
- Can be rewritten as an optimization over symmetric separable states

$$F_Q[\varrho, A] = 2 \inf_{\substack{\varrho_{\mathrm{ss}} \in S_{\mathrm{s}}, \\ \mathrm{Tr}_1(\varrho_{\mathrm{ss}})=\varrho}} \langle (A \otimes 1 - 1 \otimes A)^2\rangle_{\varrho_{\mathrm{ss}}},$$

  where S_{s} is the set of symmetric separable states.
- States in S_{s} are mixtures of symmetric product states, i.e., they are of the form

$$\sum_k p_k |\Psi_k\rangle\langle\Psi_k|^{\otimes 2}.$$

  Every symmetric separable state can be written in this form.

QFI as a convex roof used for numerics II

- Set of symmetric states with a positive definite partial transpose $S_{\mathrm{SPPT}} \supset S_{\mathrm{s}}$.
- Lower bound on the quantum Fisher information

$$F_Q[\varrho, A] \geq \inf_{\substack{\varrho_{\mathrm{SPPT}} \in S_{\mathrm{SPPT}}, \\ \mathrm{Tr}_1(\varrho_{\mathrm{SPPT}}) = \varrho}} \langle (A \otimes 1 - 1 \otimes A)^2 \rangle_{\varrho_{\mathrm{SPPT}}}.$$

  The bound can be calculated with semidefinite programming.
- It is a lower bound, not an upper bound!
- Similar ideas can be used to look for a lower bound on the QFI for given operator expectation values, or compute other convex-roof quantities (e.g., linear entropy of entanglement).

GT, T. Moroder, O. Gühne, PRL 2015;
see also M. Christandl, N. Schuch, A. Winter, PRL 2010.

Outline

Bound based on the variance

- Let us define the quantity

$$V(\varrho, A) := (\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A].$$

- It is well known that $V(\varrho, A) = 0$ for pure states.
- For states sufficiently pure $V(\varrho, A)$ is small.
- For states that are far from pure, the difference can be larger.

Generalized variance

- Generalized variances (with "old" normalization) are defined as

$$\widehat{\mathrm{var}}_{\varrho}^{f}(A) \quad = \quad \sum_{ij} m_f(\lambda_i, \lambda_j)|A_{ij}|^2 - \left(\sum \lambda_i A_{ii}\right)^2,$$

  where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a standard matrix monotone function, and $m_f(a, b) = bf(b/a)$ is a corresponding mean. Petz, J. Phys. A 35, 929 (2002);

  Gibilisco, Hiai, and Petz, IEEE Trans. Inf. Theory 55, 439 (2009).

- The $f(x)$ are bounded as

$$f_{\min}(x) \leq f(x) \leq f_{\max}(x),$$

  where

$$f_{\min}(x) = \frac{2x}{1+x}, \quad f_{\max}(x) = \frac{1+x}{2}.$$

Generalized variance II

- The generalized variance with $f(x) = f_{\max}(x)$ is the usual variance

$$\hat{\mathrm{var}}_\varrho^{\max}(A) = \langle A^2 \rangle - \langle A \rangle^2.$$

  (m_f is the arithmetic mean.)

- Let us now consider the generalized variance with $f_{\min}(x)$ $m_{\min}(a,b) = 2ab/(a+b)$. Then,

$$\hat{\mathrm{var}}_\varrho^{\min}(A) \equiv V(\varrho, A).$$

  (m_f is the harmonic mean.)

- V is the smallest among the generalized varinces

$$V(\varrho, A) = (\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A] \leq \hat{\mathrm{var}}_\varrho^f(A) \leq (\Delta A)^2.$$

Bound based on the variance, rank-2

Observation

For rank-2 states ϱ,

$$(\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A] = \frac{1}{2}[1 - \mathrm{Tr}(\varrho^2)](\tilde{\sigma}_1 - \tilde{\sigma}_2)^2$$

holds, where $\tilde{\sigma}_k$ are the nonzero eigenvalues of the matrix

$$A_{kl} = \langle k|A|l\rangle.$$

Here $|k\rangle$ are the two eigenvectors of ϱ with nonzero eigenvalues. Thus, $\tilde{\sigma}_k$ are the eigenvalues of A on the range of ϱ.

Note

$$S_{\mathrm{lin}}(\varrho) = 1 - \mathrm{Tr}(\varrho^2) = 1 - \sum_k \lambda_k^2 = \sum_{k \neq l} \lambda_k \lambda_l.$$

G. Tóth, arxiv:1701.07461.

Bound based on the variance, arbitrary rank

Observation

For states ϱ with an arbitrary rank we have

$$(\Delta A)^2 - \frac{1}{4}F_Q[\varrho, A] \leq \frac{1}{2}S_{\mathrm{lin}}(\varrho)\left[\sigma_{\max}(A) - \sigma_{\min}(A)\right]^2,$$

where $\sigma_{\max}(X)$ is the largest eigenvalue of X.

Estimate F_Q:

1. Measure the variance.
2. Estimate the purity.
3. Find a lower bound on F_Q.

G. Tóth, arxiv:1701.07461.

Bound based on the variance, arbitrary rank II

- *Proof.* V can also be defined as a concave roof

$$V(\varrho, A) = (\Delta A)^2 - \tfrac{1}{4} F_Q[\varrho, A] = \sup_{\{p_k, \Psi_k\}} \sum_k p_k (\langle A \rangle_{\Psi_k} - \langle A \rangle)^2.$$

- We want to show that

$$\sup_{\{p_k, \Psi_k\}} \sum_k p_k (\langle A \rangle_{\Psi_k} - \langle A \rangle)^2 \leq \tfrac{1}{2} S_{\rm lin}(\varrho) \left[\sigma_{\max}(A) - \sigma_{\min}(A)\right]^2.$$

- Our relation is true, if and only if

$$X := \tfrac{1}{2} S_{\rm lin}\left(\sum_k p_k |\Psi_k\rangle\langle\Psi_k|\right) \left[\sigma_{\max}(A) - \sigma_{\min}(A)\right]^2 - \sum_k p_k \left(\langle A \rangle_{\Psi_k} - \langle A \rangle\right)^2$$

is non-negative for all possible choices for p_k and $|\Psi_k\rangle$.

Bound based on the variance, arbitrary rank III

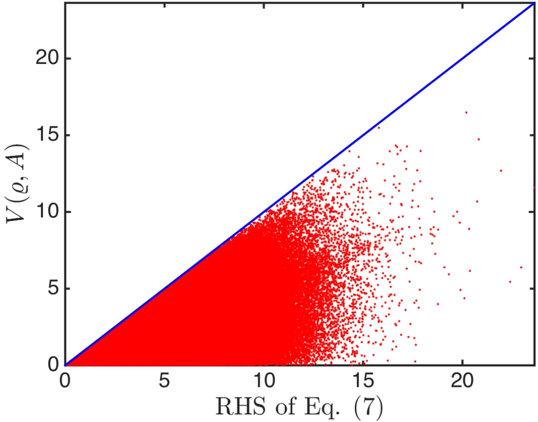
- Minimize X over $\vec{p} = (p_1, p_2, p_3, ...)$ under the constraints $p_k \geq 0$, $\sum_k p_k = 1$, while keeping the $|\Psi_k\rangle$ fixed. Further constraint: $\langle A \rangle = \sum_k p_k \langle A \rangle_{\Psi_k} = A_0$, where A_0 is a constant.

- X is a concave function of p_k's. Hence, it takes its minimum on the extreme points of the convex set of the allowed values for $\vec{p}$. For the extreme points, at most two of the p_k's are non-zero.

- Thus, we need to consider rank-2 states only, for which the statement is true due to previous observation

$$(\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A] = \frac{1}{2}[1 - \mathrm{Tr}(\varrho^2)](\tilde{\sigma}_1 - \tilde{\sigma}_2)^2.$$

Bound based on the variance, numerical test



$$(\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A] \le \frac{1}{2} S_{\text{lin}}(\varrho)[\sigma_{\max}(A) - \sigma_{\min}(A)]^2. \qquad (7)$$

Summary

- We discussed that the quantum Fisher information can be defined as a convex roof of the variance.
- We also discussed, hoe the quantum Fisher information is connected to quantum entanglement.

THANK YOU FOR YOUR ATTENTION!

Appendix

The variance as a convex roof

The variance as a convex roof

The usual variance equals four times this concave roof

$$(\Delta A)^2{}_{\varrho} = \sup_{\{p_k,\Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k},$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

The variance as a convex roof II

- Decomposition of the density matrix $\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$.
- For all decompositions $\{\tilde{p}_k, |\tilde{\Psi}_k\rangle\}$

$$(\Delta A)^2{}_{\varrho} \geq \max_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k} \geq \sum_k \tilde{p}_k (\Delta A)^2{}_{\tilde{\Psi}_k}.$$

- Important property of the variance:

$$(\Delta A)^2{}_{\varrho} = \sum_k \tilde{p}_k \left[(\Delta A)^2{}_{\tilde{\Psi}_k} + (\langle A \rangle_{\tilde{\Psi}_k} - \langle A \rangle_{\varrho})^2 \right].$$

- If for ϱ there is a decomposition $\{\tilde{p}_k, |\tilde{\Psi}_k\rangle\}$ such that the subensemble expectation values equal the expectation value for the entire ensemble (i.e., $\langle A \rangle_{\tilde{\Psi}_k} = \langle A \rangle_{\varrho}$ for all k) then

$$(\Delta A)^2{}_{\varrho} = \max_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k} = \sum_k \tilde{p}_k (\Delta A)^2{}_{\tilde{\Psi}_k}.$$

- In this case, for ϱ, the usual variance $(\Delta A)^2{}_{\varrho}$ is the concave roof of the variance.

The variance as a convex roof III

Lemma 1

For any rank-2 ϱ there is such a decompositions $\{\tilde{p}_k, |\tilde{\Psi}_k\rangle\}$.

- Eigendecomposition of the state ϱ

$$\varrho = p|\Psi_1\rangle\langle\Psi_1| + (1-p)|\Psi_2\rangle\langle\Psi_2|.$$

- We define now the family of states

$$|\Psi_\phi\rangle = \sqrt{p}|\Psi_1\rangle + \sqrt{1-p}|\Psi_2\rangle e^{i\phi}.$$

- Expectation value of the operator A

$$\langle\Psi_\phi|A|\Psi_\phi\rangle = \langle A\rangle_\varrho + 2\sqrt{p(1-p)}\mathrm{Re}\left(\langle\Psi_1|A|\Psi_2\rangle e^{i\phi}\right).$$

- Clearly, there is an angle ϕ_1 such that

$$\mathrm{Re}\left(\langle\Psi_1|A|\Psi_2\rangle e^{i\phi_1}\right) = 0.$$

  For this angle

$$\langle\Psi_\phi|A|\Psi_\phi\rangle = \langle\Psi_{\phi+\pi}|A|\Psi_{\phi+\pi}\rangle = \langle A\rangle_\varrho.$$

The variance as a convex roof IV

- In the basis of the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$, we can write the projection operators onto $|\Psi_{\phi_1}\rangle$ as

$$|\Psi_{\phi_1}\rangle\langle\Psi_{\phi_1}| = \begin{bmatrix} p & \sqrt{p(1-p)}e^{-i\phi_1} \\ \sqrt{p(1-p)}e^{+i\phi_1} & 1-p \end{bmatrix}.$$

$$|\Psi_{\phi_1+\pi}\rangle\langle\Psi_{\phi_1+\pi}| = \begin{bmatrix} p & -\sqrt{p(1-p)}e^{-i\phi_1} \\ -\sqrt{p(1-p)}e^{+i\phi_1} & 1-p \end{bmatrix}.$$

- ϱ can be decomposed as

$$\varrho = \frac{1}{2}\left(|\Psi_{\phi_1}\rangle\langle\Psi_{\phi_1}| + |\Psi_{\phi_1+\pi}\rangle\langle\Psi_{\phi_1+\pi}|\right).$$

- We proved Lemma 1.

The variance as a convex roof V

Lemma 2

Eigendecomposition of a density matrix

$$\varrho_0 = \sum_{k=1}^{r_0} \lambda_k |\Psi_k\rangle\langle\Psi_k|$$

with all $\lambda_k > 0$. Rank of the density matrix as $r(\varrho_0) = r_0$, $r_0 \geq 3$. Define A_0 as

$$A_0 = \mathrm{Tr}(A\varrho_0).$$

We claim that for any A, ϱ_0 can always be decomposed as

$$\varrho_0 = p\varrho_- + (1-p)\varrho_+,$$

such that $r(\varrho_-) < r_0$, $r(\varrho_+) < r_0$, and $\mathrm{Tr}(A\varrho_+) = \mathrm{Tr}(A\varrho_-) = A_0$.

For the proof, see GT, D. Petz, Phys. Rev. A 87, 032324 (2013).

The variance as a convex roof VI

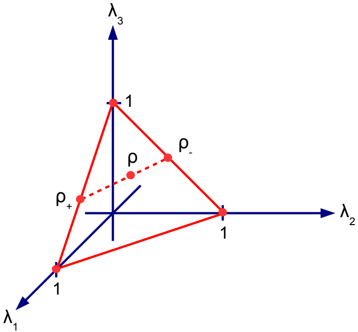


Figure: The rank-3 mixed state ϱ_0 is decomposed into the mixture of two rank-2 states, ϱ_- and ϱ_+.

- From Lemma 1 and Lemma 2, the main statement follows

$$(\Delta A)^2{}_{\varrho} = \inf_{\{p_k,\Psi_k\}} \sum_k p_k (\Delta A)^2{}_{\Psi_k}.$$

Appendix

Quantities Averaged over SU(d) generators

Quantities Averaged over SU(d) generators

- Any trcaeless Hermitian operator with $\mathrm{Tr}(A^2) = 2$ can be obtained as
$$A_{\vec{n}} := \vec{A}^T \vec{n},$$
where $\vec{A} = [A^{(1)}, A^{(2)}, A^{(3)}, ...]^T$, $\vec{n}$ is a unitvector with real elements, $(.)^T$ is matrix transpose.

- We define the average over unit vectors as
$$\overline{f} = \frac{\int f(\vec{n}) M(d\vec{n})}{\int M(d\vec{n})},$$

- We would like to compute average of V for operators.

- It is zero only for pure states. $\rightarrow$ Similar to entropies.

Bound on the average V

Observation

The average of V over traceless Hermitian matrices with a fixed norm is given as

$$\overline{V}(\varrho) = \frac{2}{d^2 - 1}\left[S_{\rm lin}(\varrho) + H(\varrho) - 1\right],$$

where d is the dimension of the system, and

$$H(\varrho) = 2\sum_{k,l}\frac{\lambda_k\lambda_l}{\lambda_k + \lambda_l} = 1 + 2\sum_{k\neq l}\frac{\lambda_k\lambda_l}{\lambda_k + \lambda_l}.$$

Average quantum Fisher information

- The average of the quantum Fisher information can be obtained as

$$\overline{F}_Q[\varrho] = \frac{8}{N_{\rm g}}[d - H(\rho)].$$

- It is maximal for pure states.

Bound based on the variance II

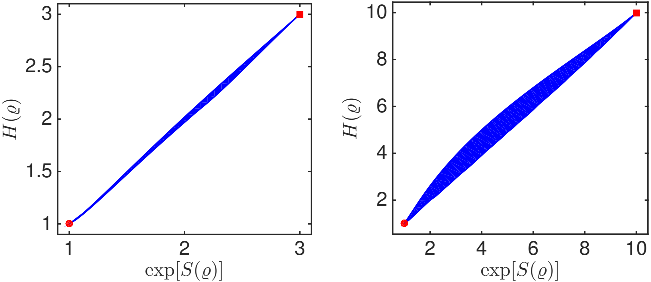


Figure: The relation between the von-Neumann entropy and $H(\varrho)$ for $d = 3$ and 10.
(filled area) Physical quantum states.
(dot) Pure states.
(square) Completely mixed state.

We see that

$$H(\varrho) \sim \exp[S(\varrho)].$$

Other type of quantum Fisher information

- The alternative form of the usual quantum Fisher information is defined as

$$\frac{d^2}{d^2\theta} S(\varrho || e^{-iA\theta} \varrho e^{+iA\theta})|_{\theta=0} = F_{\rm Q}^{\log}[\varrho, A].$$

- With that

$$\overline{F}_{\rm Q}^{\log}[\varrho] = -\frac{2}{N_{\rm g}} \left(2dS + 2 \sum_k \log \lambda_k \right).$$