Lower bounds on the quantum Fisher information based on the variance and various types of entropies

G. Tóth[1,2,3]

[1]Theoretical Physics, University of the Basque Country (UPV/EHU), Bilbao, Spain
[2]IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
[3]Wigner Research Centre for Physics, Budapest, Hungary

DFG, Erlangen
5 March 2018


F.Y.R.O.M. - The Former Yugoslav Republic of Macedonia
Greenland (DENMARK)
Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Norwegian Sea
Barents Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
SWEDEN
FINLAND
RUSSIA
Tórshavn
Faroe Islands (DENMARK)
SHETLAND ISLANDS
ORKNEY ISLANDS
HEBRIDES
Rockall (U.K.)
North Atlantic Ocean
North Sea
Celtic Sea
UNITED KINGDOM
IRELAND
Dublin
Belfast
London
DENMARK
GERMANY
POLAND
BELARUS
UKRAINE
LATVIA
LITHUANIA
ESTONIA
FRANCE
Paris
Bay of Biscay
Bilbao
A Coruña
Bordeaux
Toulouse
Zaragoza
Madrid
Lisbon
PORTUGAL
Porto
SPAIN
Barcelona
Valencia
Sevilla
Málaga
ANDORRA
ITALY
Rome
ROMANIA
BULGARIA
HUNGARY
AUSTRIA
CZECH REPUBLIC
SLOVAKIA
GREECE
Athens
MOLDOVA
Mediterranean Sea
BALEARIC ISLANDS
ALGERIA
MOROCCO
TUNISIA
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

1. **Motivation**
   - Estimating the quantum Fisher information is important
2. **Bacground**
   - Quantum Fisher information
   - Estimation of the QFI based on measurements
3. **Results**
   - Bounding the quantum Fisher information based on the variance

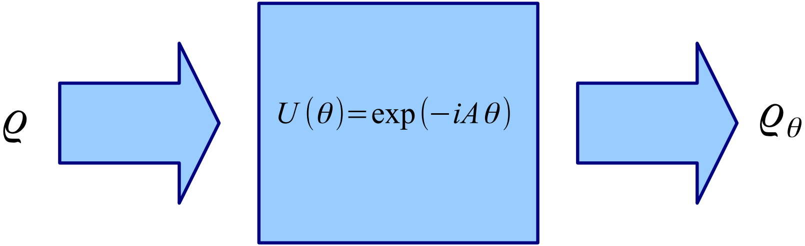
Estimating the quantum Fisher information is important

- Many experiments are aiming to carry out a metrological task.
- The quantum Fisher information tells us the best precision achievable.

Outline

Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- Large $F_Q \rightarrow$ High precision.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Properties of the Fisher information

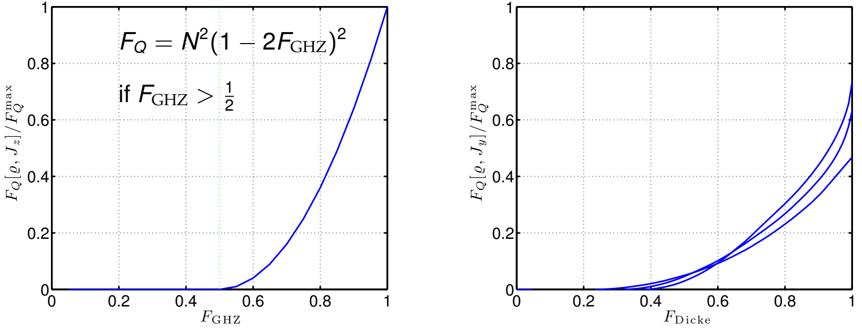
Properties of the QFI

- For pure states, it equals four times the variance, $F_Q[|\Psi\rangle\langle\Psi|, A] = 4(\Delta A)^2{}_\Psi$.
- For mixed states, $F_Q[\varrho, A] \leq 4(\Delta A)^2{}_\varrho$.
- Convex in ϱ.

Outline

Example: estimate QFI based on measurements

- Lower bound on the quantum Fisher information based on some measurements



Quantum Fisher information vs. Fidelity with respect to
(a) GHZ states and (b) Dicke states
for $N = 4, 6, 12$. $F_Q^{\max} = N^2$.

[I. Apellaniz, M. Kleinmann, O. Gühne, GT, Phys. Rev. A 2017]

Outline

1. Motivation
   - Estimating the quantum Fisher information is important
2. Bacground
   - Quantum Fisher information
   - Estimation of the QFI based on measurements
3. **Results**
   - Bounding the quantum Fisher information based on the variance

Bound based on the variance

We define the quantity

$$V(\varrho, A) := (\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A].$$

- Concave in ϱ, a generalized variance.
  [Petz, J. Phys. A 35, 929 (2002);
  Gibilisco, Hiai, and Petz, IEEE Trans. Inf. Theory 55, 439 (2009)]
- $V(\varrho, A) = 0$ for pure states.
- For states close to be pure, $V(\varrho, A)$ is small.
- For states that are far from pure, $V(\varrho, A)$ can be larger.

Bound based on the variance, rank-2

Observation 1.—For rank-2 states ϱ, the equality

$$(\Delta A)^2 - \tfrac{1}{4}F_Q[\varrho, A] = \tfrac{1}{2}[1 - \mathrm{Tr}(\varrho^2)](\tilde{\sigma}_1 - \tilde{\sigma}_2)^2$$

holds, where $\tilde{\sigma}_k$ are the nonzero eigenvalues of the matrix

$$A_{kl} = \langle k|A|l\rangle.$$

Here $|k\rangle$ are the two eigenvectors of ϱ with nonzero eigenvalues.

Note

$$S_{\mathrm{lin}}(\varrho) = 1 - \mathrm{Tr}(\varrho^2) = 1 - \sum_k \lambda_k^2 = \sum_{k \neq l} \lambda_k \lambda_l.$$

Bound based on the variance, rank-2 II

- Two-dimensional subspace

$$\{|000..00\rangle, |111..11\rangle\}.$$

- Relevant for experiments with trapped ions, creating GHZ states

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000..00\rangle + |111..11\rangle\right).$$

- The QFI is obtained as

$$F_Q[\varrho, J_l] = 2N^2\left[\mathrm{Tr}(\varrho^2) - \langle P_{000..00}\rangle^2 - \langle P_{111..11}\rangle^2\right].$$

[GHZ experiments with ione traps: D. Leibfried *et al.*, Science 2004; C. Sackett *et al.*, Nature 2000; T. Monz *et al.*, Phys. Rev. Lett. 2011.]

Bound based on the variance, arbitrary rank

Observation 2.—For states ϱ with an arbitrary rank we have

$$(\Delta A)^2 - \frac{1}{4} F_Q[\varrho, A] \leq \frac{1}{2} S_{\rm lin}(\varrho)[\sigma_{\max}(A) - \sigma_{\min}(A)]^2,$$

where $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are the largest and smallest eigenvalue of A, respectively.

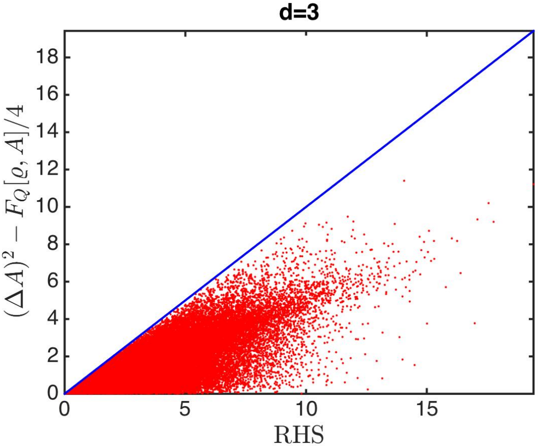
Estimate QFI:

$$F_Q[\varrho, A] \geq 4(\Delta A)^2 - 2 S_{\rm lin}(\varrho)[\sigma_{\max}(A) - \sigma_{\min}(A)]^2.$$

1. Measure the variance.
2. Estimate the purity.
3. Find a lower bound on F_Q.

Bound based on the variance, arbitrary rank II

Numerical verification of the bound



Quantities averaged over SU(d) generators

- Traceless Hermitian matrices

$$A_{\vec{n}} := \vec{A}^T \vec{n},$$

where $\vec{A} = [A^{(1)}, A^{(2)}, A^{(3)}, ...]^T$ are the SU(d) generators.

- $N_{\rm g} = d^2 - 1$ is the number of generators.

- Average over unit vectors

$$\mathrm{avg}_{\vec{n}}(\Delta A_{\vec{n}})^2 = \int (\Delta A_{\vec{n}})^2 M(d\vec{n}) = \frac{1}{N_{\rm g}} \sum_{k=1}^{N_{\rm g}} (\Delta A_k)^2.$$

Similar statement holds for $F_Q[\varrho, A]$ and $V(\varrho, A)$.

Bound on the average V

Observation 3.—Average of V over traceless Hermitian matrices

$$\mathrm{avg}_{\vec{n}} V(\varrho, A_{\vec{n}}) = \frac{2}{d^2 - 1}\left[S_{\mathrm{lin}}(\varrho) + H(\varrho) - 1\right],$$

d is the dimension of the system.

The quantity $H(\varrho)$ is defined as

$$H(\varrho) = 2\sum_{k,l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l} = 1 + 2\sum_{k \neq l} \frac{\lambda_k \lambda_l}{\lambda_k + \lambda_l}.$$

Average V is zero only for pure states. $\rightarrow$ Similar to entropies.

Average quantum Fisher information

- Average of the quantum Fisher information

$$\mathrm{avg}_{\vec{n}} F_Q[\varrho, A_{\vec{n}}] = \frac{8}{N_{\rm g}}[d - H(\rho)].$$

- Maximal for pure states.

Bound based on the QFI based on H

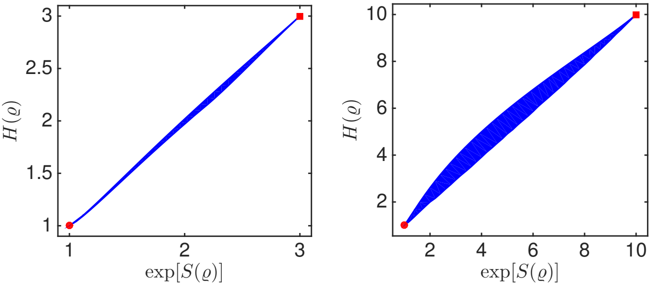


Figure: Relation of $H(\varrho)$ and the von Neumann entropy for $d = 3$ and 10.

(filled area) Physical quantum states.
(circle) Pure states.
(square) Completely mixed state.

We see that

$$H(\varrho) \sim \exp[S(\varrho)]. \qquad \text{Remember: } \mathrm{avg}\, F_Q = \frac{8}{N_{\mathrm{g}}}[d - H(\rho)].$$

Bound based on the QFI based on H

Message:

Large entropy $\rightarrow$ Small average QFI

Small entropy $\rightarrow$ Large average QFI

The more mixed the state,
the less useful it is for metrology.

Kubo-Mori-Bogoliubov quantum Fisher information I

Consider

$$F_Q^{\log}[\varrho, A] = \sum_{k,l} [\log(\lambda_k) - \log(\lambda_l)](\lambda_k - \lambda_l)|A_{kl}|^2,$$

that fulfils

$$\frac{d^2}{d^2\theta} S(\varrho || e^{-iA\theta} \varrho e^{+iA\theta})|_{\theta=0} = F_Q^{\log}[\varrho, A].$$

We found

$$\mathrm{avg}_{\vec{n}} F_Q^{\log}[\varrho, A_{\vec{n}}] = -\frac{2}{N_{\mathrm{g}}}(2dS + 2\sum_k \log \lambda_k).$$

Kubo-Mori-Bogoliubov quantum Fisher information II

- Relation to other works in the literature:

  Quantum version of the classical isoperimetric inequality relating the KMB QFI and the $\exp(S)$ for Gaussian states.

  [S. Huber, R. Koenig, and A. Vershynina, arxiv:1606.08603;
  C. Rouze, N. Datta, and Y. Pautrat, arxiv:1607.04242.]

Summary

- We discussed how to find lower bounds on the quantum Fisher information with the variance and the entropy.

See:
G. Tóth,

Lower bounds on the quantum Fisher information based on the variance and various types of entropies,
arxiv:1701.07461.

THANK YOU FOR YOUR ATTENTION!

European Research Council

MINISTERIO DE CIENCIA E INNOVACIÓN

EUSKO JAURLARITZA GOBIERNO VASCO