Criteria for detecting entanglement close to Dicke states with many-body correlations

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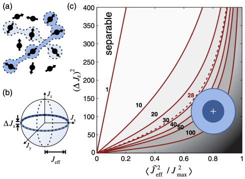
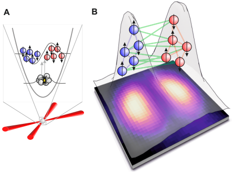
Motivation

- There have been successful experiments in detecting multipartite and bipartite entanglement in Dicke states of many particles.





Lücke PRL 2014, Vitagliano NJP 2017; Lange Science 2018, Vitagliano Quantum 2023

- They need measuring the collective observables J_x, J_y and J_z.
- The resolution of the particle number detection is not 1 particle. It can be for instance ~ 10.
- Particle-number resolving detection could improve the detected quality of the state dramatically.
- We could also have new entanglement criteria relying on single particle resolution.

Outline

Parity measurement

- We can measure the paritity as

$$\langle \sigma_z \otimes \sigma_z \otimes .... \otimes \sigma_z \rangle = \langle f(J_z) \rangle ,$$

  where

$$f(z) = e^{i2\pi(z+N/2)}.$$

- E. g, for $N = 4$, we have

$$\{f(z)\}_{z=-2,-1,0,1,2} = \{+1, -1, +1, -1, +1\}.$$

- We do not need individual access to the particles, but we need a particle number resolving detection.

Dicke states and GHZ states

- Symmetric Dicke state with $\langle J_z \rangle = 0$

$$|D_N\rangle = \binom{N}{N/2}^{-1/2} \sum_k P_k(|0\rangle^{\otimes N/2}|1\rangle^{\otimes N/2}), \tag{1}$$

where P_k denote permutations different from each other.

- GHZ state

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \tag{2}$$

- For both states

$|\langle \sigma_x \otimes \sigma_x \otimes ... \otimes \sigma_x \rangle| = |\langle \sigma_y \otimes \sigma_y \otimes ... \otimes \sigma_y \rangle| = |\langle \sigma_z \otimes \sigma_z \otimes ... \otimes \sigma_z \rangle| = 1$

for $l = x, y, z$.

Entanglement conditions with many-body correlations

For separable states

$$|\langle \sigma_x \otimes \sigma_x \otimes \ldots \otimes \sigma_x \rangle| + |\langle \sigma_y \otimes \sigma_y \otimes \ldots \otimes \sigma_y \rangle| + |\langle \sigma_z \otimes \sigma_z \otimes \ldots \otimes \sigma_z \rangle| \le 1$$

holds.

- For the ideal Dicke state the value is 3.

N	$\langle \sigma_x^{\otimes N} \rangle$	$\lvert\langle \sigma_z^{\otimes N} \rangle\rvert$	$\langle J_x^2 + J_y^2 \rangle$	$\mathcal{J}$	$(\Delta J_z)^2$
2	0.892(22)	0.965(13)	1.892(22)	0.946(11)	0.0176(66)
4	0.821(44)	0.951(25)	5.08(29)	0.85(5)	0.025(12)
6	0.833(61)	0.942(33)	11.26(85)	0.94(7)	0.029(17)
8	0.821(70)	0.806(70)	19.0(16)	0.95(8)	0.098(36)
10	0.872(72)	0.822(86)	25.7(26)	0.86(9)	0.091(45)
12	0.61(13)	0.862(96)	33.7(46)	0.80(11)	0.067(44)

Extended Data Table 1: Measurement results for various particle numbers. The uncertainties denote one standard deviation.

Proof

For separable states

$$|\langle\sigma_x\otimes\sigma_x\otimes ... \otimes\sigma_x\rangle| + |\langle\sigma_y\otimes\sigma_y\otimes ... \otimes\sigma_y\rangle| + |\langle\sigma_z\otimes\sigma_z\otimes ... \otimes\sigma_z\rangle| \leq 1$$

holds.

- *Proof.* For a product state of the type

  $$|\Psi^{(1)}\rangle\otimes|\Psi^{(2)}\rangle\otimes ... \otimes|\Psi^{(N)}\rangle$$

  the left-hand side can be bounded from above as

  $$\sum_{l=x,y,z}\left|\prod_{n=1}^{N}\left\langle\sigma_l^{(n)}\right\rangle\right| \leq \left|\left\langle\sigma_x^{(1)}\right\rangle\left\langle\sigma_x^{(2)}\right\rangle\right| + \left|\left\langle\sigma_y^{(1)}\right\rangle\left\langle\sigma_y^{(2)}\right\rangle\right| + \left|\left\langle\sigma_z^{(1)}\right\rangle\left\langle\sigma_z^{(2)}\right\rangle\right| \leq 1$$

  where in the first inequality we used that $|\langle\sigma_l^{(n)}\rangle| \leq 1$, and in the second inequality we used the Cauchy-Schwarz inequality and the fact that the length of the Bloch vector is at most one for a qubit.
- Separable states are mixtures of product states, hence the inequality is also valid for separable states. □

States detected

- The witness also detects the GHZ states as entangled.
- It also detects the singlet state given as

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes ... \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

  has

$$(\Delta J_z)^2 = 0,$$

  and

$$\left\langle \sigma_x^{\otimes N} \right\rangle = 1, \quad \left\langle \sigma_y^{\otimes N} \right\rangle = 1,$$

  if N is divisible by 4. This is a 2-entangled state.
- These operators cannot be used to detect genuine multipartite entanglement of a multi-qubit state.

Outline

Inequality with multi-particle correlations

Observation 1. For N-qubit quantum states,

$$\langle J_x \rangle^2 / j^2 + \left\langle J_y \right\rangle^2 / j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1$$

holds, where $j = N/2$ and

$$J_l = \frac{1}{2} \sum_{n=1}^{N} \sigma_l^{(n)}$$

for $l = x, y, z$.

Proof. The ground state of the Hamiltonian

$$H = BJ_x + K\sigma_z^{\otimes N},$$

where B and K are constants, is of the form

$$|\Psi\rangle = \alpha|0\rangle_x^{\otimes N} + \beta|1\rangle_x^{\otimes N},$$

which is a generalized GHZ state in the x-basis.

Inequality with multi-particle correlations II

Then, the relevant expectation value of J_x is

$$\langle J_x \rangle = \frac{N}{2} \langle \sigma_x \rangle_\phi$$

and the expectation value of the products of σ_z matrices is

$$\left\langle \sigma_z^{\otimes N} \right\rangle = \langle \sigma_z \rangle_\phi \,,$$

where we define the single-qubit state

$$|\phi\rangle = \alpha |0\rangle_x + \beta |1\rangle_x .$$

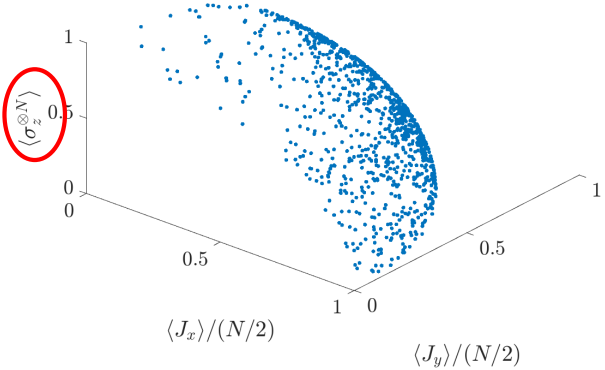
Since $\langle \sigma_x \rangle_\phi^2 + \langle \sigma_z \rangle_\phi^2 \leq 1$, it follows that

$$\langle J_x \rangle^2 / j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1.$$

Then, assuming that the mean spin is not parallel with the x-axis, but it is in the xy-plane, we arrive at our inequality. □

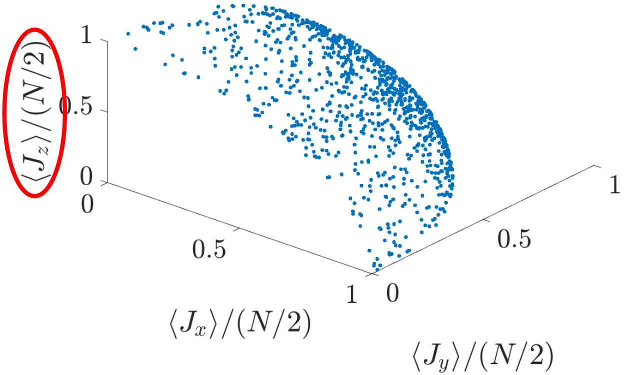
Inequality with multi-particle correlations III

Generalized GHZ states:



Inequality with multi-particle correlations IV

Comparison: spin coherent states



Bipartite conditions

Observation 2. For bipartite separable states,

$$\langle J_x \otimes J_x \rangle/(j_1 j_2) + \langle J_y \otimes J_y \rangle/(j_1 j_2) + \left|\langle \sigma_z^{\otimes N_1} \otimes \sigma_z^{\otimes N_2} \rangle\right| \leq 1$$

holds, where for the left half we have

$$j_1 = N_1/2, \qquad j_2 = N_2/2.$$

N_1 particles	N_2 particles

Proof. We start from Observation 1

$$\langle J_x \rangle^2/j^2 + \langle J_y \rangle^2/j^2 + \langle \sigma_z^{\otimes N} \rangle^2 \leq 1$$

and use the Cauchy-Schwarz inequality. □

Bipartite conditions

- Problem: we need to measure observables in the two halves of the system.
- In many experiments, we measure only collective observables.
- We need to modify the inequality such that it works for that case.
- Note that we need to measure the particle number with a single particle resolution.

Bipartite conditions

Observation 3. The following expression is true for bipartite separable states

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2),$$

where

$$j_1 = N_1/2, \qquad j_2 = N_2/2, \qquad j = N/2.$$

Proof. We start from the previous Observation. We add to both sides

$$\left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle / (2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 \right\rangle / (2j_1 j_2).$$

Then follows the relation

$$\begin{aligned} &\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \\ &\leq 1 + \left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle / (2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 / (2j_1 j_2). \right\rangle \end{aligned}$$

Bipartite conditions II

Then, starting from the relation

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \\ \leq 1 + \left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle / (2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 \right\rangle / (2j_1 j_2),$$

we use the inequality

$$\left\langle (J_x^{(n)})^2 + (J_y^{(n)})^2 \right\rangle \leq j_n (j_n + 1).$$

We arrive at

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2).$$

We need to measure only collective quantities! □

Outline

1. Motivation
2. Detection of multipartite and bipartite entanglement close to Dicke states
3. **Criteria with many-body correlations**
   - Simple criterion with many-body correlations
   - Bipartite criterion
   - Multiparticle entanglement

Conditions for multi-particle entanglement

Observation 4. States violating the inequality

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2),$$

for

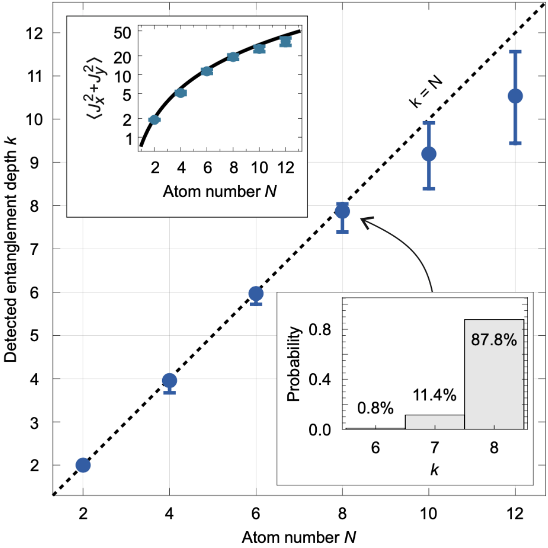
$$j_1 = k/2, \qquad j_2 = (N-k)/2$$

k particles	$N-k$ particles

possess at least $(k+1)$-particle entanglement, where we assume that $k \geq N/2$.

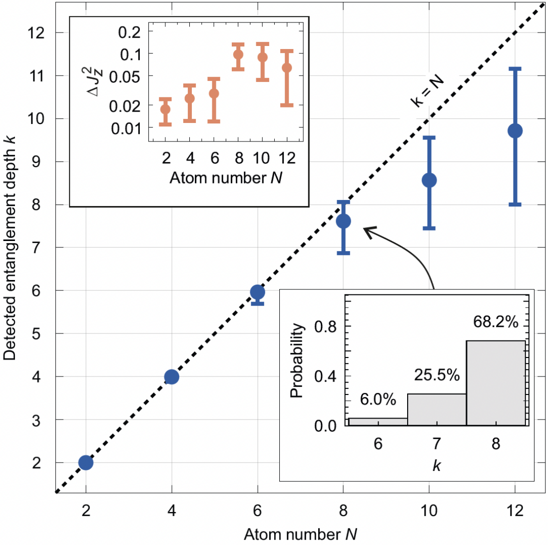
Violation for $k = N - 1$ means genuine multipartite entanglement.

Results



Detected entanglement depth k
Atom number N
k = N
$\langle J_x^2 + J_y^2 \rangle$
Atom number N
Probability
87.8%
11.4%
0.8%
k

Comparison to the alternative method



Conclusions

- We discussed how to detect bipartite and multipartite entanglement with many-body correlation measurements.
- The method has been successfully used in experiments with Dicke states up to 12 particles.
- It demonstrates the good quality of the created Dicke state.
- For the transparencies, see

  www.gtoth.eu

- See also

M. Quensen, M. Hetzel, L. Santos, A. Smerzi,
G. Tóth, L. Pezzé, C. Klempt.

Hong-Ou-Mandel interference of more than 10 indistinguishable atoms,

arXiv:2504.02691.

THANK YOU FOR YOUR ATTENTION!