Witnessing metrologically useful multiparticle entanglement

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Why multipartite entanglement and metrology are important?

- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.
- We should tell
    - How entangled the state is
    - What the state is good for, etc.

Outline

Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_k^{(1)} \otimes \varrho_k^{(2)} \otimes ... \otimes \varrho_k^{(N)}.$$

If a state is not separable then it is entangled (Werner, 1989).

k-producibility/k-entanglement

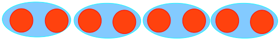
A pure state is *k-producible* if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

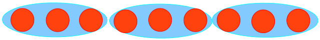
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., Gühne, GT, NJP 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.



two-producible



three-producible

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ are Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

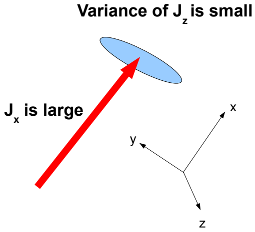
The standard spin-squeezing criterion

Spin squeezing criteria for entanglement detection

$$\xi_s^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

If $\xi_s^2 < 1$ then the state is entangled. [Sørensen, Duan, Cirac, Zoller, Nature (2001).]

- States detected are like this:



Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- A full set of generalized spin squeezing criteria is known for the case above.

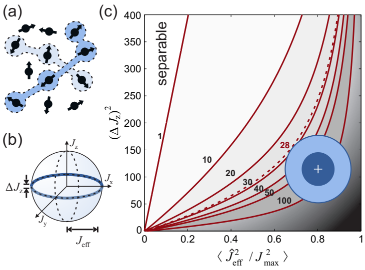
[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

[Higher spins: G. Vitagliano, P. Hyllus, I. Egusquiza, GT, Phys. Rev. Lett. 2011]

[Experiments with singlets: Behbood *et al.,* Phys. Rev. Lett. 2014;
GT, Mitchell, New. J. Phys. 2010.]

Multipartite entanglement detection with spin squeezing (only two criteria!)

- Original spin-squeezing method
  [Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, Nature 464, 1165 (2010).]
- Generalized method. BEC, 8000 particles.
  28-particle entanglement is detected.



[Lücke *et al.*, Phys. Rev. Lett. 112, 155304 (2014), also in Synopsys in physics.aps.org.]

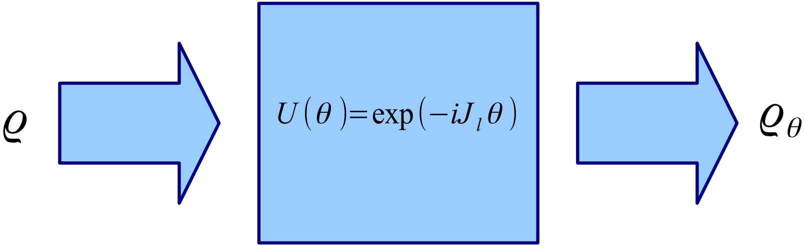
Outline

Our main goals

- Detect metrologically useful multipartite entanglement, not just entanglement in general.
- Detect multipartite entanglement in the vicinity of various states.

Quantum metrology

- Fundamental task in metrology with a linear interferometer



- We have to estimate θ in the dynamics

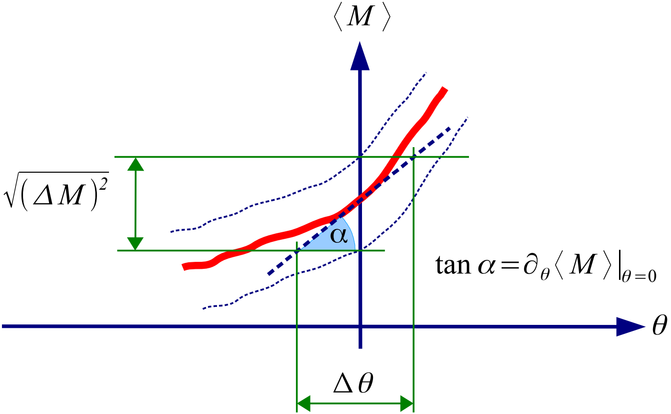
$$U = \exp(-iJ_l\theta)$$

where $l \in \{x, y, z\}$.

Precision of parameter estimation

- Measure an operator M to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad \frac{1}{(\Delta\theta)^2} \leq F_Q[\varrho, A].$$

  where $F_Q[\varrho, A]$ is the quantum Fisher information.
  (A is now the Hamiltonian of the unitary dynamics.)

- The quantum Fisher information is given by an explicit formula for ϱ and A.

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

  where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \le N.$$

[Pezze, Smerzi, PRL 2009; Hyllus, Gühne, Smerzi, PRA 2010]

- For states with at most k-particle entanglement (k is divisor of N)

$$F_Q[\varrho, J_l] \le kN.$$

[Hyllus *et al.*, PRA 2012; GT, PRA 2012].

- If a state violates the above inequality then it has $(k+1)$-particle metrologically useful entanglement.

Connection to GHZ states

If a state has $(k+1)$-particle metrologically useful entanglement then it performs better than k-particle GHZ states, i. e., $|\mathrm{GHZ}_k\rangle^{\otimes \frac{N}{k}}$.

k-particle Greenberger-Horne-Zeilinger state ("Scrödinger cat" state)

$$|\mathrm{GHZ}_k\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes k} + |1\rangle^{\otimes k}).$$

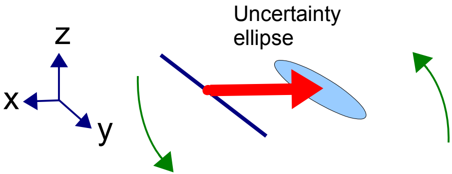
Outline

Metrology with spin-squeezed states

- Pezze-Smerzi bound

$$(\Delta\theta)^2 = \frac{(\Delta J_z)^2}{|\partial_\theta \langle J_z\rangle|^2} = \frac{(\Delta J_z)^2}{\langle J_x\rangle^2} = \frac{\xi_s^2}{N}.$$

- We measure $\langle J_z\rangle$.



[Pezze, Smerzi, PRL 2009.]

Outline

Dicke states

- Symmetric Dicke states with $\langle J_z \rangle = 0$ (simply "Dicke states" in the following) are defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

[photons: Kiesel, Schmid, GT, Solano, Weinfurter, PRL 2007;
Prevedel, Cronenberg, Tame, Paternostro, Walther, Kim, Zeilinger, PRL 2007;
Wieczorek, Krischek, Kiesel, Michelberger, GT, and Weinfurter, PRL 2009]

[cold atoms: Lücke *et al.*, Science 2011; Hamley *et al.*, Science 2011; C. Gross *et al.*, Nature 2011]

Metrology with Dicke states

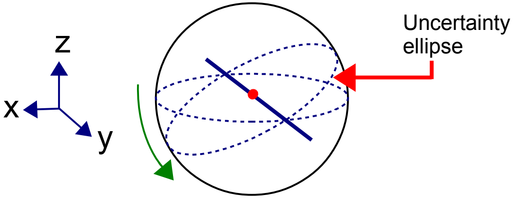
- For our symmetric Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

$$U = \exp(-iJ_y\theta).$$

- Measure $\langle J_z^2 \rangle$ to estimate θ. (We cannot measure first moments, since they are zero.)



Outline

Witnessing metrological usefulness

- Direct measurement of the sensitivity
  - Measure $(\Delta\theta)^2$.
  - Obtain bound on F_Q and multipartite entanglement, $F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)^2}$.
  - Experimentally challenging, since we need quantum dynamics.
  - The precision is affected by the noise during the dynamics.

  [Experiments in cold atoms by the groups of M. Oberthaler, C. Klempt; photonic experiments of the Weinfurter group.]

- Witnessing (our choice)
  - Estimate how good the precision were, if we did the metrological process.
  - Assume a perfect metrological process. Characterizes the state only.

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

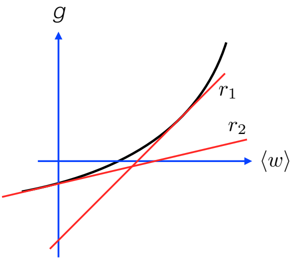
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013);
GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Legendre transform

- Optimal linear lower bound on a convex function $g(\varrho)$ based on an operator expectation value $w = \langle W \rangle_\varrho = \mathrm{Tr}(W\varrho)$



$$g(\varrho) \geq rw - const.,$$

where $w = \mathrm{Tr}(\varrho W)$.

- For every slope r there is a "*const.*"

- Textbooks say

$$g(\varrho) \geq \mathcal{B}(w) := rw - \hat{g}(rW),$$

where $\hat{g}$ is the Legendre transform

$$\hat{g}(W) = \sup_{\varrho}[\langle W \rangle_\varrho - g(\varrho)].$$

[Gühne, Reimpell, Werner, PRL 2007; Eisert, Brandao, Audenaert, NJP 2007.]

Legendre transform II

- Bound is best if we optimize over r as

$$g(\varrho) \geq \mathcal{B}(w) := \sup_r \left[rw - \hat{g}\left(rW\right)\right],$$

where again $w = \mathrm{Tr}(\varrho W)$.

- F_Q is the convex roof of the variance. Hence, it is sufficient to carry out an optimization over pure states

$$\hat{g}(W) = \sup_\Psi [\langle W \rangle_\Psi - g(\Psi)].$$

- Similar simplification has been used for entanglement measures.

[Gühne, Reimpell, Werner, PRL 2007; Eisert, Brandao, Audenaert, NJP 2007.]

Legendre transform III

- For our case, the Legendre transform is

$$\hat{\mathcal{F}}_{\mathrm{Q}}(W) = \sup_{\Psi}[\langle W - 4J_l^2\rangle_\Psi + 4\,\langle J_l\rangle_\Psi^2].$$

- With further simplifications, an optimization over a single real variable is needed

$$\hat{\mathcal{F}}_{\mathrm{Q}}(W) = \sup_{\mu}\left\{\lambda_{\max}\left[W - 4(J_l - \mu)^2\right]\right\}.$$

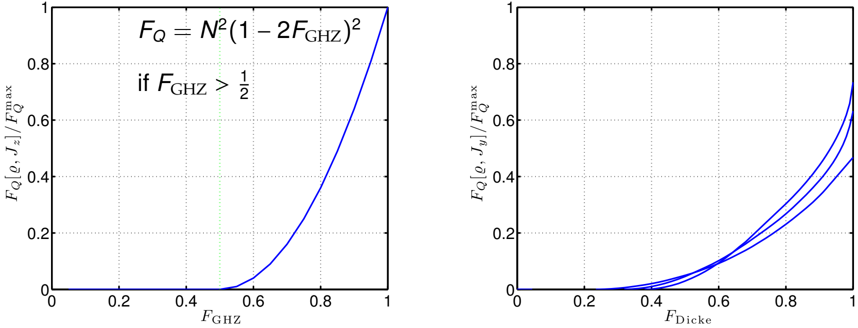
Legendre transform IV

Big surprise

The quantum Fisher information is the ideal quantity for using the Legendre transform technique.

Witnessing the quantum Fisher information based on the fidelity

- Let us bound the quantum Fisher information based on some measurements. First, consider small systems.
  [See also Augusiak *et al.*, 1506.08837.]



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

[Apellaniz *et al.*, PRA 95, 032330 (2017).]

Bounding the qFi based on collective measurements

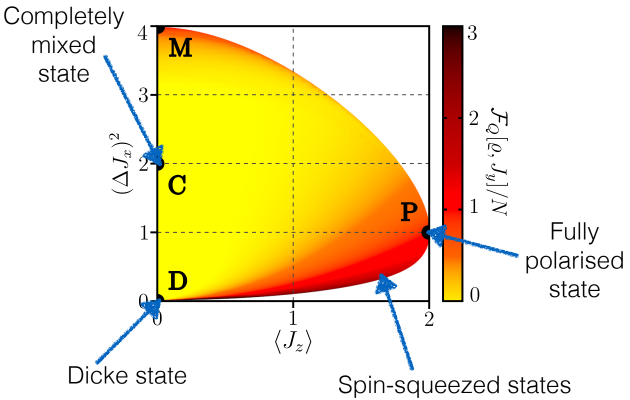
Bound for the quantum Fisher information for spin squeezed states (Pezze-Smerzi bound)

$$F_Q[\varrho, J_y] \geq \frac{\langle J_z \rangle^2}{(\Delta J_x)^2}.$$

[Pezze, Smerzi, PRL 2009.]

Bounding the qFi based on collective measurements II

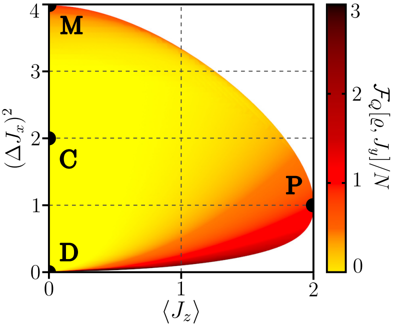
- Optimal bound for the quantum Fisher information $F_Q[\varrho, J_y]$ for spin squeezing for $N = 4$ particles



[Apellaniz, Kleinmann, Gühne, GT, PRA 95, 032330 (2017).]

Bounding the qFi based on collective measurements III

- Optimal bound for the quantum Fisher information $F_Q[\varrho, J_y]$ for spin squeezing for $N = 4$ particles

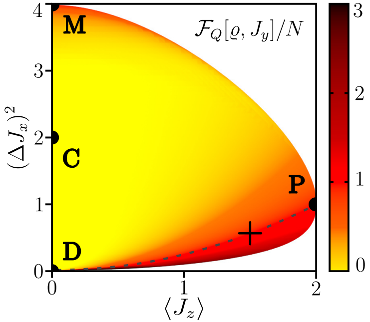


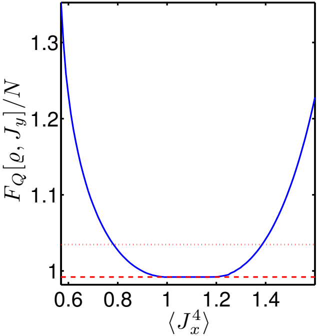
On the bottom part of the figure $[(\Delta J_x)^2 < 1]$ the bound is very close to the Pezze-Smerzi bound!

[Apellaniz, Kleinman, Gühne, GT, PRA 95, 032330 (2017).]

Bounding the qFi based on collective measurements IV

- The bound can be obtained if additional expectation value, i.e., $\langle J_x^2 \rangle$ is measured, or we assume symmetry:





[Apellaniz, Kleinman, Gühne, GT, PRA 95, 032330 (2017).]

Spin squeezing experiment

- Experiment with $N = 2300$ atoms,

$$\xi_s^2 = -8.2\mathrm{dB} = 10^{-8.2/10} = 0.1514.$$

[Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 2010.]

- The Pezze-Smerzi bound is:

$$\frac{\mathcal{F}_Q[\varrho_N, J_y]}{N} \geq \frac{1}{\xi_s^2} = 6.605.$$

- We get the same value for our method!

[Pezze, Smerzi, PRL 2009]

- Similar calculations for Dicke state experiments!

[Lücke, Peise, Vitagliano, Arlt, Santos, GT, Klempt, PRL 2014.]

Ongoing work

- Lower bound on the quantum Fisher information with the variance and the purity

$$(\Delta J_l)^2 - \tfrac{1}{4} F_Q[\varrho, J_l] \leq \frac{N^2}{2}[1 - \mathrm{Tr}(\varrho^2)].$$

[G. Tóth, arXiv:1701.07461.]

Summary

- We discussed a very flexible method to detect multipartite entanglement and metrological usefulness.
- We can choose a set of operators and the method gives an optimal lower bound on F_Q.

Apellaniz, Lücke, Peise, Klempt, GT, New J. Phys. 17, 083027 (2015);

Apellaniz, Kleinmann, Gühne, GT, Phys. Rev. A 95, 032330 (2017), Editors' Suggestion.

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