Criteria for detecting entanglement close to Dicke states with many-body correlations

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Quantum entanglement in qubits and CV systems,
EHU, Leioa,
October 14, 2025.

Outline

Entanglement - Pure states

- Q: What is entanglement for pure states?
- A: bipartite state can be a product state $|\Psi_A\rangle \otimes |\Psi_B\rangle$, or an entangled state.
- For instance, $|00\rangle$ and $|11\rangle$ are product states.
- $(|00\rangle + |11\rangle)/\sqrt{2}$ is an entangled state.
- We can always decide whether a pure state is entangled.

Entanglement - Mixed states

Definition

A quantum state is called separable if it can be written as a convex sum of product states as [Werner, 1989]

$$\varrho = \sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)},$$

where p_k form a probability distribution ($p_k > 0, \sum_k p_k = 1$), and $\varrho_n^{(k)}$ are single-qudit density matrices.

A state that is not separable is called entangled.

- We cannot always decide whether the state is entangled.

k-producibility/k-entanglement

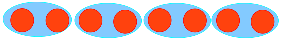
A pure state is k-producible if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

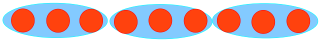
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
e.g., Gühne, GT, NJP 2005.

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

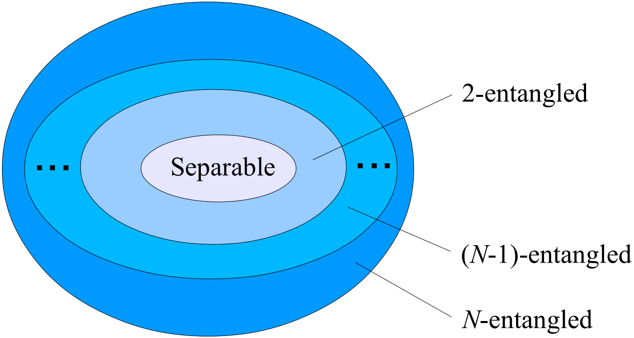


2-entangled



3-entangled

k-producibility/k-entanglement II



$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$ 2-entangled

$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$ 3-entangled

$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle)$ 4-entangled

Outline

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.

- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

Summing over all permutations.
Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

Spin Squeezing Inequality for Dicke states

- For separable states

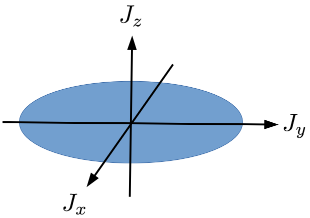
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

holds. GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007

- It detects entangled states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \mathrm{max.},$$
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Multipartite entanglement - Dicke states

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x\rangle^2 + \langle J_y\rangle^2}}{J_{\max}}\right).$$

- Combine it with

$$\langle J_x^2 + J_y^2\rangle \leqslant J_{\max}(\tfrac{k}{2}+1) + \langle J_x\rangle^2 + \langle J_y\rangle^2,$$

which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x^2 + J_y^2\rangle - J_{\max}(\frac{k}{2}+1)}}{J_{\max}}\right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

G. Vitagliano *et al.*, NJP 2017.

Outline

Dicke state of cold atoms

PRL 112, 155304 (2014) PHYSICAL REVIEW LETTERS week ending 18 APRIL 2014

Detecting Multiparticle Entanglement of Dicke States

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(Received 27 February 2014; published 17 April 2014)

Recent experiments demonstrate the production of many thousands of neutral atoms entangled in their spin degrees of freedom. We present a criterion for estimating the amount of entanglement based on a measurement of the global spin. It outperforms previous criteria and applies to a wider class of entangled states, including Dicke states. Experimentally, we produce a Dicke-like state using spin dynamics in a Bose-Einstein condensate. Our criterion proves that it contains at least genuine 28-particle entanglement. We infer a generalized squeezing parameter of $-11.4(5)$ dB.

DOI: 10.1103/PhysRevLett.112.155304 PACS numbers: 67.85.–d, 03.67.Bg, 03.67.Mn, 03.75.Mn

Entanglement, one of the most intriguing features of quantum mechanics, is nowadays a key ingredient for many applications in quantum information science [1,2], quantum simulation [3,4], and quantum-enhanced metrology [5]. Entangled states with a large number of particles cannot be characterized via full state tomography [6], which is routinely used in the case of photons [7,8], trapped ions [9], or superconducting circuits [10,11]. A reconstruction of the full density matrix is hindered and finally prevented by the exponential increase of the required number of measurements. Furthermore, it is technically impossible to address all individual particles or even fundamentally forbidden if the particles occupy the same quantum state. Therefore, the entanglement of many-particle states is best characterized by measuring the expectation values and variances of the components of the collective spin $\mathbf{J} = (J_x, J_y, J_z)^T = \sum_i \mathbf{s}_i$, the sum of all individual spins $\mathbf{s}_i$ in the ensemble.

In particular, the spin-squeezing parameter $\xi^2 = N(\Delta J_z)^2/(\langle J_x\rangle^2 + \langle J_y\rangle^2)$ defines the class of spin-squeezed states for $\xi^2 < 1$. This inequality can be used to verify the presence of entanglement, since all spin-squeezed states are entangled [12]. Large clouds of entangled neutral atoms are typically prepared in such spin-squeezed states, as shown in thermal gas cells [13], at ultracold temperatures [14–16], and in Bose-Einstein

quantified by means of the so-called entanglement depth, defined as the number of particles in the largest nonseparable subset [see Fig. 1(a)]. There have been numerous experiments detecting multiparticle entanglement involving up to 14 qubits in systems, where the particles can be addressed individually [9,20–24]. Large ensembles of neutral atoms

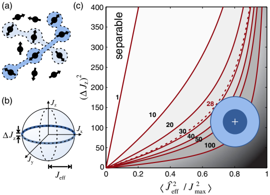
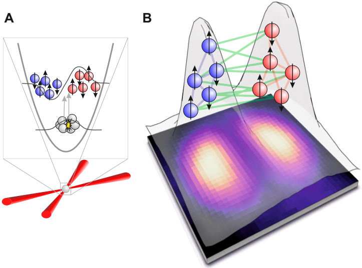


FIG. 1 (color online). Measurement of the entanglement depth for a total number of 8000 atoms. (a) The entanglement depth is given by the number of atoms in the largest nonseparable subset

Outline

Experiment in cold gases

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



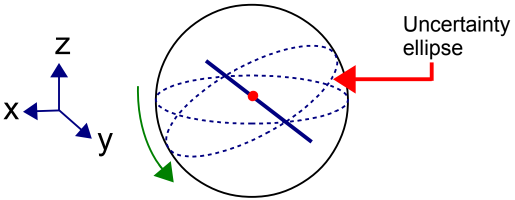
K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

Symmetric Dicke state

- For our symmetric Dicke state

$$\begin{aligned}\langle J_x\rangle = \langle J_y\rangle = \langle J_z\rangle &= 0,\\ \langle J_x^2\rangle = \langle J_y^2\rangle &= \text{large},\\ \langle J_z^2\rangle &= 0.\end{aligned}$$

- Pancake-like uncertainty ellipse, we can even rotate it with an external field



Correlations for Dicke states

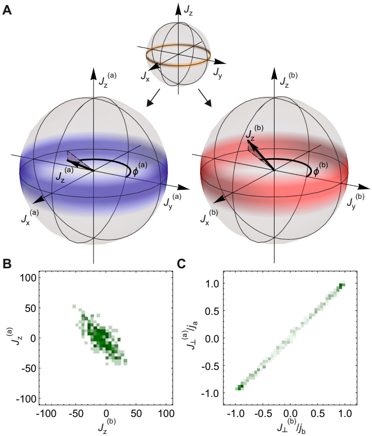
- For the Dicke state

$$
\begin{aligned}
(\Delta(J_x^a - J_x^b))^2 &\approx 0, \\
(\Delta(J_y^a - J_y^b))^2 &\approx 0, \\
(\Delta J_z)^2 = (\Delta(J_z^a + J_z^b))^2 &= 0.
\end{aligned}
$$

- Measurement results on well "*b*" can be predicted from measurements on "*a*"

$$
\begin{aligned}
J_x^b &\approx J_x^a, && \text{(correlation)} \\
J_y^b &\approx J_y^a, && \text{(correlation)} \\
J_z^b &= -J_z^a. && \text{(anti-correlation)}
\end{aligned}
$$

Correlations for Dicke states - experimental results



Here, $J_\perp^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

Outline

The two-well entanglement criterion

Main result

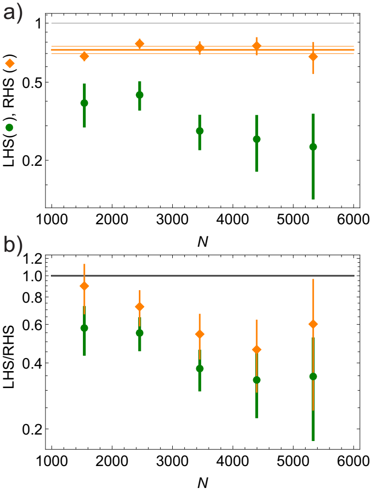
For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta(J_x^a - J_x^b))^2 + (\Delta(J_y^a - J_y^b))^2\right] \geq \frac{\left\langle J_x^2 + J_y^2 \right\rangle^2}{N(N+2)}$$

holds. $|D_N\rangle : \frac{1}{4}$ $\frac{N}{4}$ $\frac{N(N+2)}{16}$

Similar criterion for EPR steering.

Violation of the criterion: entanglement is detected II



LHS/RHS for similar, but somewhat more complicated inequalities. (top) Quantum 2024, and (bottom) for Science 2018.

Bipartite entanglement detection

Other experiments creating bipartite entanglement in BEC, published back-to-back in 2018:

Spatially separated parts of a spin-squeezed Bose-Einstein condensate, two-component condensate:

M. Fadel, T. Zibold, B. Décamps, and P. Treutlein,
Spatial entanglement patterns and Einstein-Podolsky-Rosen steering in Bose-Einstein condensates,
Science 360, 409 (2018).

Spatially separated parts of a spin-squeezed Bose-Einstein condensate, spin-1 particles.

P. Kunkel, M. Prüfer, H. Strobel, D. Linnemann, A. Frölian, T. Gasenzer, M. Gärttner, and M. K. Oberthaler,
Spatially distributed multipartite entanglement enables EPR steering of atomic clouds,
Science 360, 413 (2018).

Outline

Particle number resolving detection

- The resolution of the particle number detection is not 1 particle. Typically, ~ 10.
- So far we did not need single particle resolution.
- Particle-number resolving detection could improve the detected quality of the state dramatically.
- We could also have new entanglement criteria relying on single particle resolution.
- It is possible to reach a single-particle resolution:

M. Quensen, M. Hetzel, L. Santos, A. Smerzi, G. Tóth, L. Pezzé, C. Klempt, Hong-Ou-Mandel interference of more than 10 indistinguishable atoms, arXiv:2504.02691.

Parity measurement

- We can measure the parity as

$$\langle \sigma_z \otimes \sigma_z \otimes .... \otimes \sigma_z \rangle = \langle f(J_z) \rangle,$$

where

$$f(z) = e^{i2\pi(z+N/2)}.$$

- E. g, for $N = 4$, we have

$$\{f(z)\}_{z=-2,-1,0,1,2} = \{+1, -1, +1, -1, +1\}.$$

- Thus, we do not need individual access to the particles, but we need a particle number resolving detection.

Entanglement conditions with many-body correlations

- For separable states

$$\left|\langle\sigma_x\otimes\sigma_x\otimes\ldots\otimes\sigma_x\rangle\right|+\left|\langle\sigma_y\otimes\sigma_y\otimes\ldots\otimes\sigma_y\rangle\right|+\left|\langle\sigma_z\otimes\sigma_z\otimes\ldots\otimes\sigma_z\rangle\right|\leq 1$$

  holds.

- For the ideal Dicke state the value is 3.

N	$\langle\sigma_x^{\otimes N}\rangle$	$\lvert\langle\sigma_z^{\otimes N}\rangle\rvert$	$\langle J_x^2+J_y^2\rangle$	$\mathcal{J}$	$(\Delta J_z)^2$
2	0.892(22)	0.965(13)	1.892(22)	0.946(11)	0.0176(66)
4	0.821(44)	0.951(25)	5.08(29)	0.85(5)	0.025(12)
6	0.833(61)	0.942(33)	11.26(85)	0.94(7)	0.029(17)
8	0.821(70)	0.806(70)	19.0(16)	0.95(8)	0.098(36)
10	0.872(72)	0.822(86)	25.7(26)	0.86(9)	0.091(45)
12	0.61(13)	0.862(96)	33.7(46)	0.80(11)	0.067(44)

Extended Data Table 1: Measurement results for various particle numbers. The uncertainties denote one standard deviation.

Proof

For separable states

$$\left|\langle\sigma_x\otimes\sigma_x\otimes...\otimes\sigma_x\rangle\right|+\left|\langle\sigma_y\otimes\sigma_y\otimes...\otimes\sigma_y\rangle\right|+\left|\langle\sigma_z\otimes\sigma_z\otimes...\otimes\sigma_z\rangle\right|\leq 1$$

holds.

- *Proof.* For a product state of the type

  $$|\Psi^{(1)}\rangle\otimes|\Psi^{(2)}\rangle\otimes...\otimes|\Psi^{(N)}\rangle$$

  the left-hand side can be bounded from above as

  $$\sum_{l=x,y,z}\left|\prod_{n=1}^{N}\left\langle\sigma_l^{(n)}\right\rangle\right|\leq\left|\left\langle\sigma_x^{(1)}\right\rangle\left\langle\sigma_x^{(2)}\right\rangle\right|+\left|\left\langle\sigma_y^{(1)}\right\rangle\left\langle\sigma_y^{(2)}\right\rangle\right|+\left|\left\langle\sigma_z^{(1)}\right\rangle\left\langle\sigma_z^{(2)}\right\rangle\right|\leq 1$$

  where in the first inequality we used that $\left|\left\langle\sigma_l^{(n)}\right\rangle\right|\leq 1$, and in the second inequality we used the Cauchy-Schwarz inequality and the fact that the length of the Bloch vector is at most one for a qubit.
- Separable states are mixtures of product states, hence the inequality is also valid for separable states. □

States detected

- The witness also detects the GHZ states as entangled.
- The singlet state given as

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes ... \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

  has

$$(\Delta J_z)^2 = 0,$$

  and

$$\left\langle \sigma_x^{\otimes N} \right\rangle = 1, \quad \left\langle \sigma_y^{\otimes N} \right\rangle = 1,$$

  if N is divisible by 4.

- Thus, these operators cannot be used to detect genuine multipartite entanglement.

Inequality with multi-particle correlations

Observation 1. For N-qubit quantum states,

$$\langle J_x \rangle^2 / j^2 + \left\langle J_y \right\rangle^2 / j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1$$

holds, where $j = N/2$ and

$$J_l = \frac{1}{2} \sum_{n=1}^{N} \sigma_l^{(n)}$$

for $l = x, y, z$.

Proof. The ground state of the Hamiltonian

$$H = BJ_x + K\sigma_z^{\otimes N},$$

where B and K are constants, is of the form

$$|\Psi\rangle = \alpha|0\rangle_x^{\otimes N} + \beta|1\rangle_x^{\otimes N},$$

which is a generalized Greenberger-Horne-Zeilinger (GHZ) state in the x-basis.

Inequality with multi-particle correlations II

Then, the relevant expectation value of J_x is

$$\langle J_x \rangle = \frac{N}{2} \langle \sigma_x \rangle_\phi$$

and the expectation value of the products of σ_z matrices is

$$\left\langle \sigma_z^{\otimes N} \right\rangle = \langle \sigma_z \rangle_\phi \,,$$

where we define the single-qubit state

$$|\phi\rangle = \alpha|0\rangle_x + \beta|1\rangle_x.$$

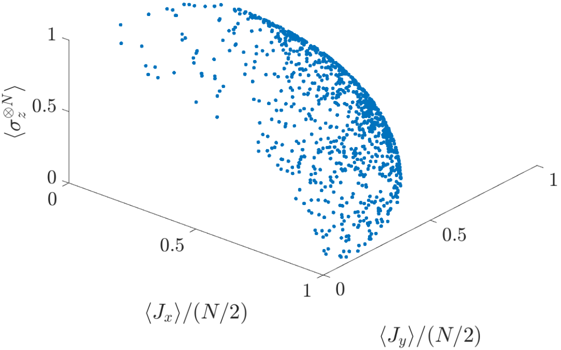
Since $\langle \sigma_x \rangle_\phi^2 + \langle \sigma_z \rangle_\phi^2 \leq 1$, it follows that

$$\langle J_x \rangle^2 / j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1.$$

Then, assuming that the mean spin is not in the x-direction, but is in the xy-plane, we arrive at our inequality. $\square$

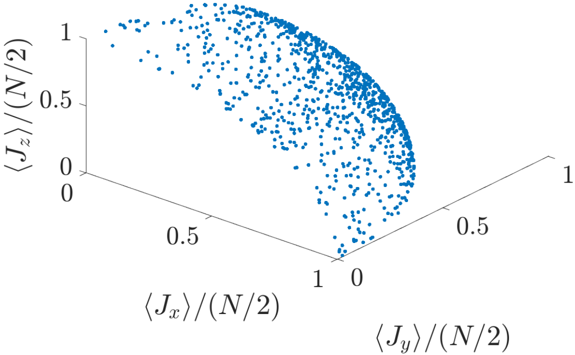
Inequality with multi-particle correlations III

Generalized GHZ states:



Inequality with multi-particle correlations IV

Comparison: spin coherent states



Bipartite conditions

Observation 2. For bipartite separable states,

$$\langle J_x \otimes J_x \rangle /(j_1 j_2) + \left\langle J_y \otimes J_y \right\rangle /(j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N_1} \otimes \sigma_z^{\otimes N_2} \right\rangle \right| \leq 1$$

holds, where for the left half we have

$$j_1 = N_1/2, \qquad j_2 = N_2/2.$$

N_1 particles	N_2 particles

Proof. We start from Observation 1

$$\langle J_x \rangle^2 /j^2 + \left\langle J_y \right\rangle^2 /j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1$$

and use the Cauchy-Schwarz inequality. □

Bipartite conditions

- Problem: we need to measure observables in the two halves of the system.
- In many experiments, we measure only collective observables.
- We need to modify the inequality such that it works for that case.
- Note that we need to measure the particle number with a single particle resolution.

Bipartite conditions

Observation 3. The following expression is true for bipartite separable states

$$\sum_{l=x,y} \left\langle (J_l^{(1)} + J_l^{(2)})^2 \right\rangle /(2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2),$$

where

$$j_1 = N_1/2, \qquad j_2 = N_2/2, \qquad j = N/2.$$

Proof. We start from the previous Observation. We add to both sides

$$\sum_{l=x,y} \left\langle (J_l^{(1)})^2 \right\rangle /(2j_1 j_2) + \left\langle (J_l^{(2)})^2 \right\rangle /(2j_1 j_2).$$

Then follows the relation

$$\begin{aligned}&\sum_{l=x,y} \left\langle (J_l^{(1)} + J_l^{(2)})^2 \right\rangle /(2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \\ &\leq 1 + \sum_{l=x,y} \left\langle (J_l^{(1)})^2 \right\rangle /(2j_1 j_2) + \left\langle (J_l^{(2)})^2 \right\rangle /(2j_1 j_2)\end{aligned}$$

Bipartite conditions II

Then, starting from the relation

$$\sum_{l=x,y} \left\langle (J_l^{(1)} + J_l^{(2)})^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right|$$
$$\leq 1 + \sum_{l=x,y} \left\langle (J_l^{(1)})^2 \right\rangle / (2j_1 j_2) + \left\langle (J_l^{(2)})^2 \right\rangle / (2j_1 j_2),$$

we use the inequality

$$\left\langle (J_x^{(n)})^2 + (J_y^{(n)})^2 \right\rangle \leq j_n(j_n + 1).$$

We arrive at

$$\sum_{l=x,y} \left\langle (J_l^{(1)} + J_l^{(2)})^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2).$$

We need to measure only collective quantities! $\square$

Outline

Conditions for multi-particle entanglement

Observation 4. States violating the inequality

$$\sum_{l=x,y} \left\langle (J_l^{(1)} + J_l^{(2)})^2 \right\rangle / (2j_1 j_2) + |\left\langle \sigma_z^{\otimes N} \right\rangle| \leq j(j+1)/(2j_1 j_2),$$

for

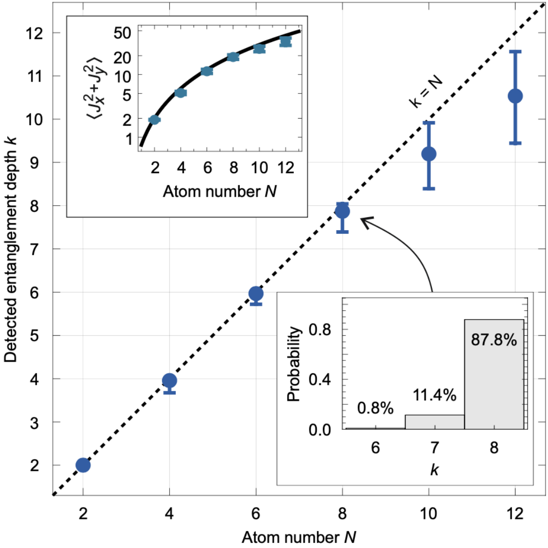
$$j_1 = k/2, \qquad j_2 = (N-k)/2$$

k particles	$N-k$ particles

possess at least $(k+1)$-particle entanglement, where we assume that $k \geq N/2$.

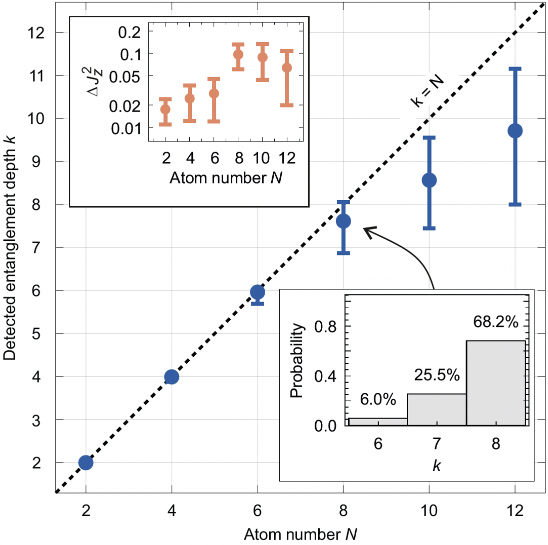
Violation for $k = N-1$ means genuine multipartite entanglement.

Results



Detected entanglement depth k
Atom number N
$k = N$
$\langle J_x^2 + J_y^2 \rangle$
Atom number N
Probability
87.8%
11.4%
0.8%
k

Comparison



Detected entanglement depth k
Atom number N
k = N
ΔJ_z^2
0.2
0.1
0.05
0.02
0.01
2 4 6 8 10 12
Atom number N
Probability
0.8
0.4
0.0
6.0%
25.5%
68.2%
6 7 8
k

Conclusions

- We discussed how to detect bipartite and multipartite entanglement with many-body correlation measurements.
- The method has been successfully used in experiments with Dicke states up to 12 particles.
- It demonstrates the good quality of the created Dicke state.
- For the transparencies, see

  www.gtoth.eu

- See also

  M. Quensen, M. Hetzel, L. Santos,A. Smerzi,
  G. Tóth, L. Pezzé, C. Klempt.

  Hong-Ou-Mandel interference of more than 10 indistinguishable atoms,

  arXiv:2504.02691.

THANK YOU FOR YOUR ATTENTION!