Quantum states with a positive partial transpose are useful for metrology

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Outline

1. **Motivation**
   - What are entangled states useful for?

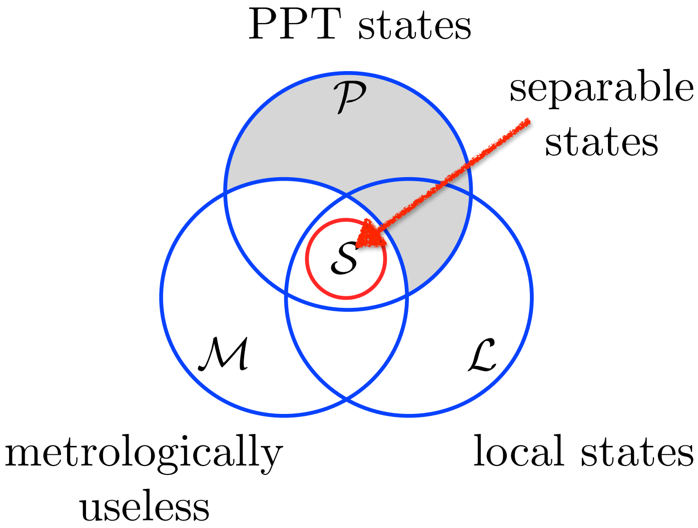
2. **Bacground**
   - Quantum Fisher information
   - Recent findings on the quantum Fisher information

3. **Maximizing the QFI for PPT states**
   - Results so far
   - Our results

What are entangled states useful for?

- Entangled states are useful, but not all of them are useful for some task.
- Entanglement is needed for beating the shot-noise limit in quantum metrology.
- Intriguing question: Are states with a positive partial transpose useful for metrology? Can they also beat the shot-noise limit?

What are entangled states useful for?



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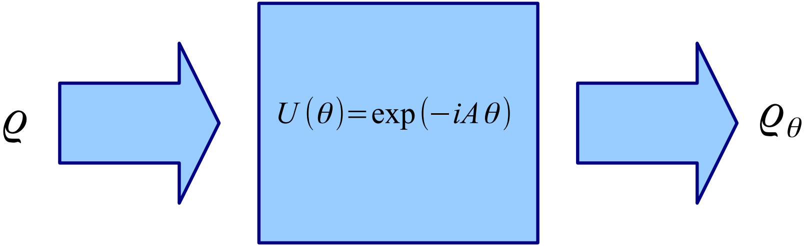
Quantum metrology

- Fundamental task in metrology



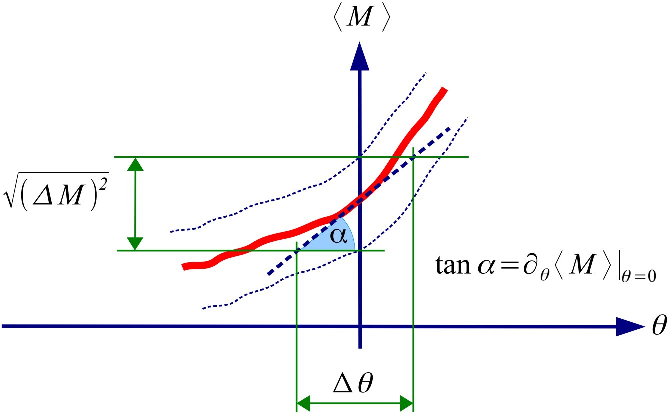
- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

Precision of parameter estimation

- Measure an operator *M* to get the estimate θ. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M\rangle|^2}.$$



The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

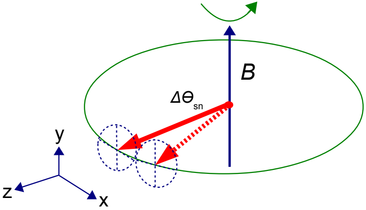
where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

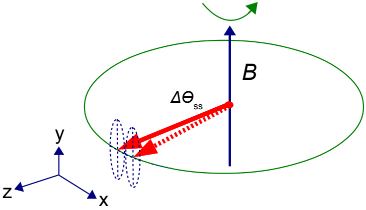
Special case $A = J_l$

- The operator A is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer





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Properties of the Fisher information

Many bounds on the quantum Fisher information can be derived from these simple properties:

- For pure states, it equals four times the variance, $F[|\Psi\rangle\langle\Psi|, A] = 4(\Delta A)^2{}_{\Psi}$.
- For mixed states, it is convex.

The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most *k*-particle entanglement (*k* is divisor of *N*)

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2,$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

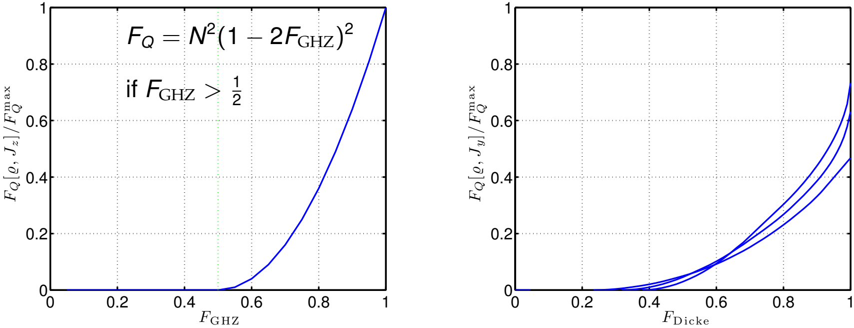
$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

Witnessing the quantum Fisher information based on few measurements

- Let us bound the quantum Fisher information based on some measurements.



Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

[Apellaniz *et al.*, Phys. Rev. A 2017]

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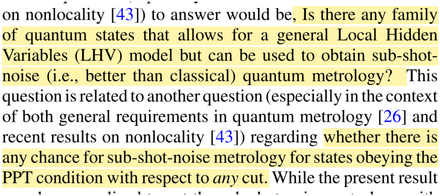
Results so far concerning metrologically useful PPT states

- Bound entangled states with PPT and some non-PPT partitions.
- Violates an entanglement criterion with three QFI terms.

  [P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, and A. Smerzi, PRA 85, 022321 (2012).]

- Non-unlockable bound entangled states with PPT and some non-PPT partitions.
- Violates the criterion with a single QFI term, better than shot-noise limit.

  [Ł. Czekaj, A. Przysiężna, M. Horodecki, P. Horodecki, Phys. Rev. A 92, 062303 (2015).]



on nonlocality [43]) to answer would be, Is there any family of quantum states that allows for a general Local Hidden Variables (LHV) model but can be used to obtain sub-shot-noise (i.e., better than classical) quantum metrology? This question is related to another question (especially in the context of both general requirements in quantum metrology [26] and recent results on nonlocality [43]) regarding whether there is any chance for sub-shot-noise metrology for states obeying the PPT condition with respect to *any* cut. While the present result

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Our results

We look for bipartite PPT entangled states and multipartite states that are PPT with respect to all partitions.

Maximizing the QFI for PPT states: brute force

- Maximize the QFI for PPT states. Remember

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- Difficult to maximize a convex function over a convex set. The maximum is taken on the boundary of the set.

- Not guaranteed to find the global maximum.

- Note: Finding the *minimum* is possible!

Maximizing the QFI for PPT state: our method

- We mentioned that the QFI gives a bound on the precision of the parameter estimation

$$F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)^2} = \frac{|\partial_\theta \langle M \rangle|^2}{(\Delta M)^2} = \frac{\langle i[M,A] \rangle^2}{(\Delta M)^2} \quad \text{(dynamics is } U = e^{-iA\theta}\text{).}$$

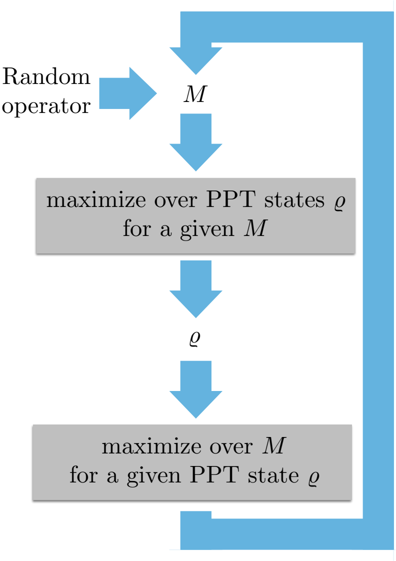
- The bound is sharp

$$F_Q[\varrho, A] = \max_M \frac{\langle i[M,A] \rangle_\varrho^2}{(\Delta M)^2}.$$

The maximum for PPT states can be obtained as

$$\max_{\varrho \text{ is PPT}} F_Q[\varrho, A] = \max_{\varrho \text{ is PPT}} \max_M \frac{\langle i[M,A] \rangle_\varrho^2}{(\Delta M)^2}.$$

Sew-saw algorithm for maximizing the precision



Random operator

M

maximize over PPT states ϱ for a given M

ϱ

maximize over M for a given PPT state ϱ

Maximize over PPT states for a given M

Best precision for PPT states for a given operator M can be obtained by a semidefinite program.

Proof.—Let us define first

$$\begin{aligned} f_M(X,Y) = \min_{\varrho} \quad & \mathrm{Tr}(M^2\varrho), \\ \text{s.t.} \quad & \varrho \geq 0, \varrho^{\mathrm{T}k} \geq 0 \text{ for all } k, \mathrm{Tr}(\varrho) = 1, \\ & \langle i[M,A] \rangle = X \text{ and } \langle M \rangle = Y. \end{aligned}$$

The best precsion for a given M and for PPT states is

$$(\Delta\theta)^2 = \min_{X,Y} \frac{f_M(X,Y) - Y^2}{X^2}.$$

The state giving the best precision is $\varrho_{\mathrm{PPTopt}}$.

Maximize over M for a given PPT state

For a state ϱ, the best precision is obtained with the operator given by the symmetric logarithmic derivative

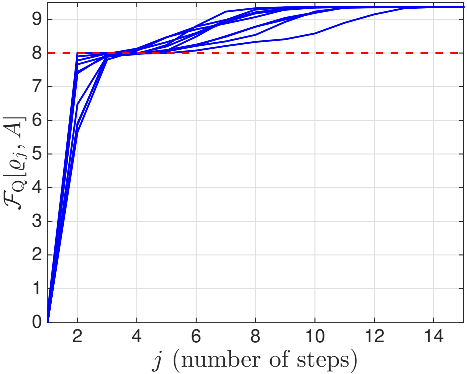
$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|A|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Convergence of the method

The precision cannot get worse with the iteration!

Convergence of the method II



Generation of the 4×4 bound entangled state.

(blue) 10 attempts. After 15 steps, the algorithm converged.

(red) Maximal quantum Fisher information for separable states.

Robustness of the states

$$\varrho(p) = (1-p)\varrho + p\varrho_{\rm noise}$$

- Robustness of entanglement: the maximal p for which $\varrho(p)$ is entangled for any separable $\varrho_{\rm noise}$.
  [Vidal and Tarrach, PRA 59, 141 (1999).]

- Robustness of metrological usefulness: the maximal p for which $\varrho(p)$ outperforms separable state for any separable $\varrho_{\rm noise}$.

Robustness of the states II

System	A	$\mathcal{F}_{\rm Q}[\varrho, A]$	$\mathcal{F}_{\rm Q}^{\rm (sep)}$	$p_{\rm white\,noise}$
four qubits	J_z	4.0088	4	0.0011
three qubits	$j_z^{(1)} + j_z^{(2)}$	2.0021	2	0.0005
2×4 (three qubits, only $1:23$ is PPT)	$j_z^{(1)} + j_z^{(2)}$	2.0033	2	0.0008

Multiqubit states

Robustness of the states III

d	$\mathcal{F}_{\rm Q}[\varrho, A]$	$p_{\rm white\,noise}$	$p_{\rm noise}^{\rm LB}$
3	8.0085	0.0006	0.0003
4	9.3726	0.0817	0.0382
5	9.3764	0.0960	0.0361
6	10.1436	0.1236	0.0560
7	10.1455	0.1377	0.0086
8	10.6667	0.1504	0.0670
9	10.6675	0.1631	0.0367
10	11.0557	0.1695	0.0747
11	11.0563	0.1807	0.0065
12	11.3616	0.1840	0.0808

- $d \times d$ systems.
- Maximum of the quantum Fisher information for separable states is 8.
- The operator A is not the usual J_z.

Robustness of the states IV: 4×4 bound entangled PPT state

Let us define the following six states
$|\Psi_1\rangle = (|0,1\rangle + |2,3\rangle)/\sqrt{2}$, $|\Psi_2\rangle = (|1,0\rangle + |3,2\rangle)/\sqrt{2}$,
$|\Psi_3\rangle = (|1,1\rangle + |2,2\rangle)/\sqrt{2}$, $|\Psi_4\rangle = (|0,0\rangle + |3,3\rangle)/\sqrt{2}$,
$|\Psi_5\rangle = (1/2)(|0,3\rangle + |1,2\rangle) + |2,1\rangle/\sqrt{2}$,
$|\Psi_6\rangle = (1/2)(-|0,3\rangle + |1,2\rangle) + |3,0\rangle/\sqrt{2}$.

Our state is a mixture

$$\varrho_{4\times 4} = p \sum_{n=1}^{4} |\Psi_n\rangle\langle\Psi_n| + q \sum_{n=5}^{6} |\Psi_n\rangle\langle\Psi_n|,$$

where $q = (\sqrt{2} - 1)/2$ and $p = (1 - 2q)/4$. We consider the operator

$$A = H \otimes \mathbb{1} + \mathbb{1} \otimes H,$$

where $H = \mathrm{diag}(1, 1, -1, -1)$.

Negativity

Apart from making calculations for PPT bound entangled states, we can also make calculations for states with given minimal eigenvalues of the partial transpose, or for a given negativity.

[G. Vidal and R. F. Werner, PRA 65, 032314 (2002).]

Entanglement

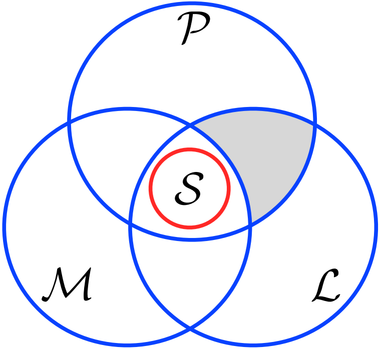
Bipartite state	Entanglement
3×3	0.0003
4×4	0.0147
5×5	0.0239
6×6	0.0359
7×7	0.0785
UPB 3×3	0.0652
Breuer 4×4	0.1150

Convex roof of the linear entanglement entropy. The entanglement is also shown for the 3×3 state based on unextendible product bases (UPB) and for the Breuer state with a parameter $\lambda = 1/6$.

[G. Tóth, T. Moroder, and O. Gühne, PRL 114, 160501 (2015).]

Metrologically useful quantum states with LHV models (PPT)

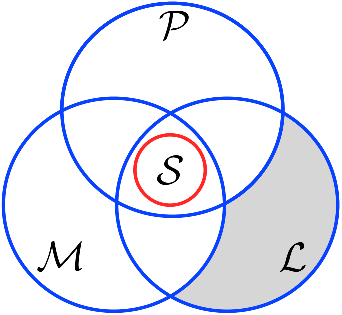
Consider the 2×4 state listed before. Possible to construct numerically a LHV model for the state.



[F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, PRL 2016; D. Cavalcanti, L. Guerini, R. Rabelo, and P. Skrzypczyk, PRL 2016.]

Metrologically useful quantum states with LHV models (non-PPT)

- Two-qubit Werner state $p|\Psi^-\rangle\langle\Psi^-| + (1-p)\mathbb{1}/4$, with $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.
- Better for metrology than separable states ($\mathcal{F}_{\rm Q} > 2$) for $p > 1 - 0.3596 = 0.6404$.
- They do not violate a Bell inequality for $p < 0.6829$.



[F. Hirsch, M. T. Quintino, T. Vértesi, M. Navascués, N. Brunner, Quantum 2017; A. Acín, N. Gisin, B. Toner, PRA 2006.]

Cluster states

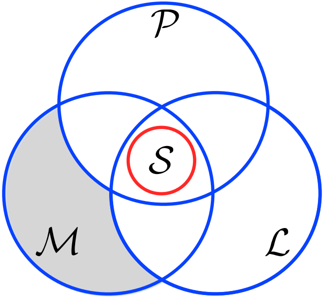
Cluster states: resource in measurement-based quantum computing

[R. Raussendorf and H. J. Briegel, PRL 2001.]

- Fully entangled pure states.
- Violate a Bell inequality

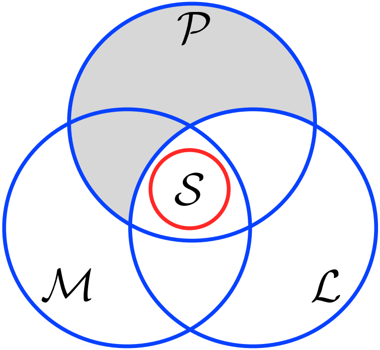
  [V. Scarani, A. Acín, E. Schenck, M. Aspelmeyer, PRA 2005; O. Gühne, GT, P. Hyllus, H. J. Briegel, PRL 2005; GT, O. Gühne, and H. J. Briegel, PRA 2006.]
- Metrologically not useful

  [P. Hyllus, O. Gühne, and A. Smerzi, PRA 2010.]



Non-local PPT states

Counterexample for the Peres conjecture



[T. Vértesi and N. Brunner, Nature Communications 2015.]

Summary

- We presented quantum states with a positive partial transpose with respect to all bipartitions that are useful for metrology.

See:

Géza Tóth and Tamás Vértesi,

Quantum states with a positive partial transpose
are useful for metrology,

arxiv:1709.03995.

THANK YOU FOR YOUR ATTENTION!

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