# Criteria for detecting entanglement close to Dicke states with many-body correlations

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University of Hannover,
Hannover,
December 11, 2025.

# Outline

# Entanglement - Pure states

- Q: What is entanglement for pure states?
- A: bipartite state can be a product state $|\Psi_A\rangle \otimes |\Psi_B\rangle$, or an entangled state.
- For instance, $|00\rangle$ and $|11\rangle$ are product states.
- $(|00\rangle + |11\rangle)/\sqrt{2}$ is an entangled state.
- We can always decide whether a pure state is entangled.

# Entanglement - Mixed states

## Definition

A quantum state is called separable if it can be written as a convex sum of product states as [Werner, 1989]

$$\varrho = \sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)},$$

where $p_k$ form a probability distribution ($p_k > 0, \sum_k p_k = 1$), and $\varrho_n^{(k)}$ are single-qudit density matrices.

A state that is not separable is called entangled.

- We cannot always decide whether the state is entangled.

# $k$-producibility/$k$-entanglement

**A pure state is $k$-producible if it can be written as**

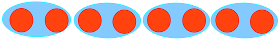
$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

where $|\Phi_l\rangle$ are states of at most $k$ qubits.

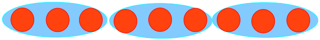
A mixed state is $k$-producible, if it is a mixture of $k$-producible pure states.

e.g., Gühne, GT, NJP 2005.

- If a state is not $k$-producible, then it is at least $(k+1)$-particle entangled.

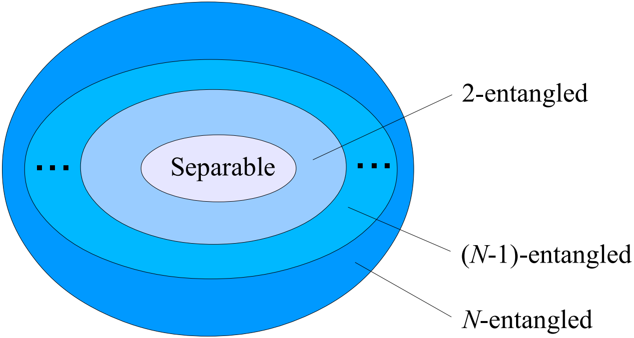


2-entangled



3-entangled

# $k$-particle entanglement




$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$ 2-entangled

$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$ 3-entangled

$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle)$ 4-entangled

# Outline

# Many-particle systems

- For spin-$\frac{1}{2}$ particles, we can measure the expectation value of the collective angular momentum operators

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variance

$$(\Delta J_l)^2 := \langle J_l^2\rangle - \langle J_l\rangle^2$$

variances.

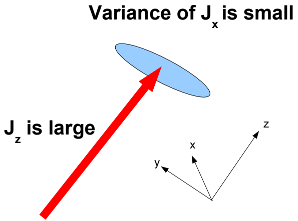
# Spin squeezing

## Definition

Uncertainty relation for the spin coordinates

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2.$$

If $(\Delta J_x)^2$ is smaller than the standard quantum limit $\frac{1}{2}|\langle J_z\rangle|$ then the state is called spin squeezed (mean spin in the $z$ direction!).
M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993)



# Spin squeezing

## Definition

Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

- Application: Quantum metrology, magnetometry. Used many times in experiments.
- Violation of the spin-squeezing entanglement criterion
  $\Rightarrow$ violation of the spin Kitagawa-Ueda criterion

A. Sørensen *et al.*, Nature **409**, 63 (2001)

# Multipartite entanglement in spin squeezing

- We consider pure $k$-producible states of the form

$$|\Psi\rangle = \otimes_{l=1}^{M} |\psi_l\rangle,$$

where $|\psi_l\rangle$ is the state of at most $k$ qubits.

## Extreme spin squeezing

The spin-squeezing criterion for $k$-producible states is

$$(\Delta J_x)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left( \frac{\sqrt{\langle J_y \rangle^2 + \langle J_z \rangle^2}}{J_{\max}} \right),$$

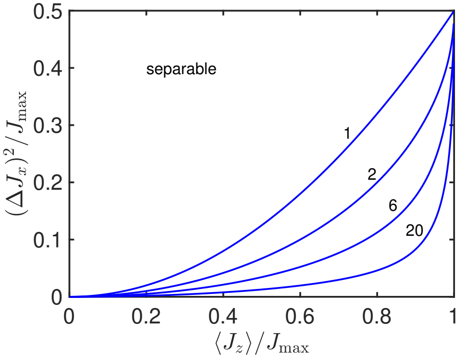
where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(Z) := \frac{1}{j} \min_{\frac{\langle j_z \rangle}{j} = Z} (\Delta j_x)^2.$$

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

# Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").




- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

# Outline

# Complete set of the generalized spin squeezing criteria

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle),$$

$$\vec{K} := (\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle).$$

- Then any state violating the following inequalities is entangled

$$\begin{aligned}
\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle &\leq N(N+2)/4, && \text{(always true)}\\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq N/2, && \text{(singlet)}\\
\langle J_k^2\rangle + \langle J_l^2\rangle - N/2 &\leq (N-1)(\Delta J_m)^2, && \text{(Dicke state)}\\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2\rangle + N(N-2)/4, && \text{(planar sq. state)}
\end{aligned}$$

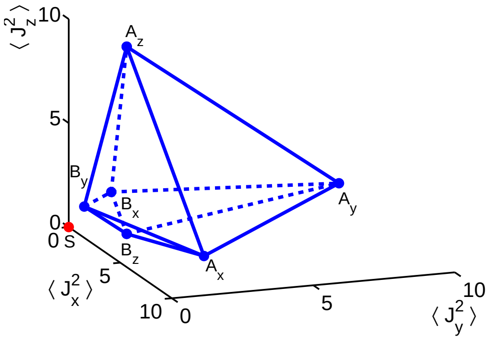
where $k, l, m$ takes all the possible permutations of $x, y, z$.

GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007.

Recent general results for spin-$j$ particles: Vitagliano *et al.*, Quantum 2025.

# The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!



# Spin squeezing criteria – Two-particle correlations

All quantities needed can be obtained with two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{\rm 2p}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{\rm 2p}}.$$

- Here, the average 2-particle density matrix is defined as

$$\varrho_{\rm 2p} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- Still, we can detect states with a separable $\varrho_{\rm 2p}$.
- Still, as we will see, we can even detect multipartite entanglement!

# Singlet state

- Singlet states are ground states of antiferromagnetic Hamiltonians with $\left\langle J_l^2 \right\rangle = 0$ for $l = x, y, z$.

- The permutationally invariant singlet is

$$\varrho_{\text{singlet}} \propto \lim_{T \to 0} e^{-\frac{J_x^2 + J_y^2 + J_z^2}{T}}.$$

- For such a state, for large $N$ we have

$$\varrho_{2\text{p}} \approx \frac{\mathbb{1}}{4},$$

  still it is detected as entangled by our criterion!

- Such a state has been created in cold atoms.

# Entanglement conditions based on the two-body density matrix

- Spin squeezing conditions with collective variables based on the two-body density matrix.
- All detected states have an entangled two-qubit density matrix, violating the PPT criterion.

Spin Squeezing Inequalities and Entanglement of *N* Qubit States,
J. K. Korbicz, J. I. Cirac, and M. Lewenstein,
Phys. Rev. Lett. 95, 120502 (2005).

# Dicke states

- Dicke states: simultaneous eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and $J_z$ [Dicke, 1954]

$$J^2|j, j_z, \alpha\rangle = j(j+1)|j, j_z, \alpha\rangle,$$
$$J_z|j, j_z, \alpha\rangle = j_z|j, j_z, \alpha\rangle.$$

- Symmetric Dicke states of spin-1/2 particles have

$$j = N/2.$$

For such states

$$\langle J_x^2 + J_y^2 + J_z^2\rangle = \frac{N}{2}\left(\frac{N}{2} + 1\right).$$

- We are interested in the state for which

$$j_z = 0.$$

For such states $\langle J_z\rangle = 0$. $(\Delta J_z)^2 = 0$, it is an eigenstate of $J_z$.

# Dicke states II

- The state is given as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left( |0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

where are summing over all permutations.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left( |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

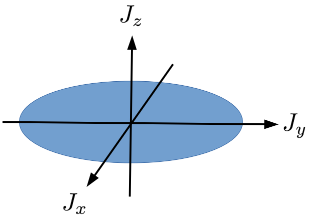
cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

# Spin Squeezing Inequality for Dicke states

- It detects entangled states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \text{maximal},$$
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.




- For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

holds.

# Multipartite entanglement - Dicke states

Condition for entanglement detection around Dicke states. For states with at most $k$-particle entanglement

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left( \frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2}+1)}}{J_{\max}} \right)$$

holds.

G. Vitagliano *et al.*, New J. Phys. 19, 013027 (2017).

# Outline

# Dicke state of cold atoms

PRL 112, 155304 (2014) PHYSICAL REVIEW LETTERS week ending 18 APRIL 2014

## Detecting Multiparticle Entanglement of Dicke States

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(Received 27 February 2014; published 17 April 2014)

Recent experiments demonstrate the production of many thousands of neutral atoms entangled in their spin degrees of freedom. We present a criterion for estimating the amount of entanglement based on a measurement of the global spin. It outperforms previous criteria and applies to a wider class of entangled states, including Dicke states. Experimentally, we produce a Dicke-like state using spin dynamics in a Bose-Einstein condensate. Our criterion proves that it contains at least genuine 28-particle entanglement. We infer a generalized squeezing parameter of −11.4(5) dB.

DOI: 10.1103/PhysRevLett.112.155304 PACS numbers: 67.85.−d, 03.67.Bg, 03.67.Mn, 03.75.Mn

Entanglement, one of the most intriguing features of quantum mechanics, is nowadays a key ingredient for many applications in quantum information science [1,2], quantum simulation [3,4], and quantum-enhanced metrology [5]. Entangled states with a large number of particles cannot be characterized via full state tomography [6], which is routinely used in the case of photons [7,8], trapped ions [9], or superconducting circuits [10,11]. A reconstruction of the full density matrix is hindered and finally prevented by the exponential increase of the required number of measurements. Furthermore, it is technically impossible to address all individual particles or even fundamentally forbidden if the particles occupy the same quantum state. Therefore, the entanglement of many-particle states is best characterized by measuring the expectation values and variances of the components of the collective spin $\mathbf{J} = (J_x, J_y, J_z)^T = \sum_i \mathbf{s}_i$, the sum of all individual spins $\mathbf{s}_i$ in the ensemble.

In particular, the spin-squeezing parameter $\xi^2 = N(\Delta J_z)^2/(\langle J_x\rangle^2 + \langle J_y\rangle^2)$ defines the class of spin-squeezed states for $\xi^2 < 1$. This inequality can be used to verify the presence of entanglement, since all spin-squeezed states are entangled [12]. Large clouds of entangled neutral atoms are typically prepared in such spin-squeezed states, as shown in thermal gas cells [13], at ultracold temperatures [14–16], and in Bose-Einstein

quantified by means of the so-called entanglement depth, defined as the number of particles in the largest nonseparable subset [see Fig. 1(a)]. There have been numerous experiments detecting multiparticle entanglement involving up to 14 qubits in systems, where the particles can be addressed individually [9,20–24]. Large ensembles of neutral atoms

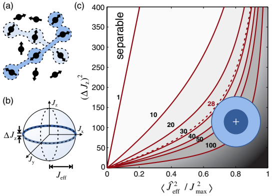



FIG. 1 (color online). Measurement of the entanglement depth for a total number of 8000 atoms. (a) The entanglement depth is given by the number of atoms in the largest nonseparable subset

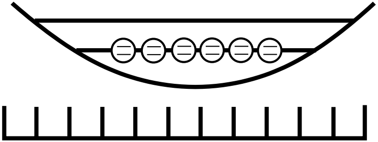
# Outline

# Bipartite entanglement from bosonic multipartite entanglement

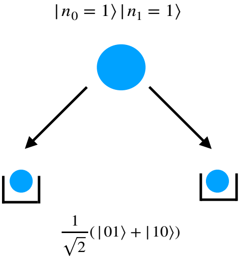
- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

# Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument



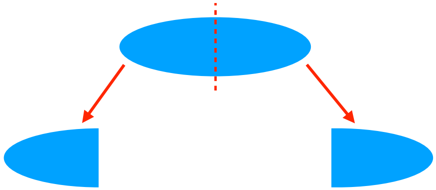
See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)



# Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.

- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.

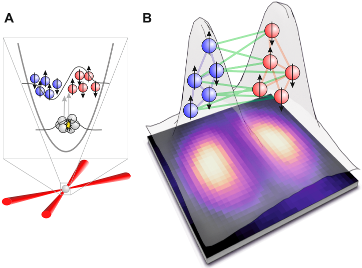


N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

# Outline

# Experiment in cold gases

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.




K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

# Correlations for Dicke states

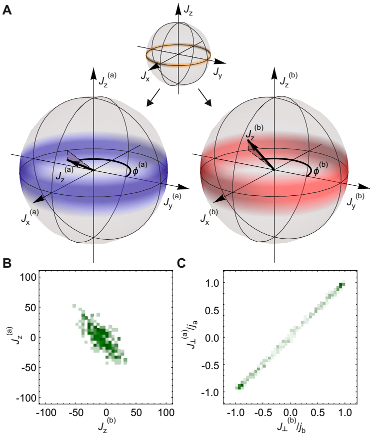
- For the Dicke state

$$
\begin{aligned}
(\Delta(J_x^a - J_x^b))^2 &\approx 0, \\
(\Delta(J_y^a - J_y^b))^2 &\approx 0, \\
(\Delta J_z)^2 = (\Delta(J_z^a + J_z^b))^2 &= 0.
\end{aligned}
$$

- Measurement results on well "*b*" can be predicted from measurements on "*a*"

$$
\begin{aligned}
J_x^b &\approx J_x^a, && \text{(correlation)} \\
J_y^b &\approx J_y^a, && \text{(correlation)} \\
J_z^b &= -J_z^a. && \text{(anti-correlation)}
\end{aligned}
$$

# Correlations for Dicke states - experimental results




$$\text{Here, } J_{\perp}^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}.$$

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

# Outline

# The two-well entanglement criterion

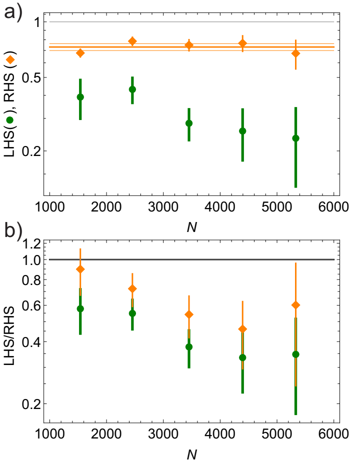
For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta(J_x^a - J_x^b))^2 + (\Delta(J_y^a - J_y^b))^2\right] \geq \frac{\left\langle J_x^2 + J_y^2 \right\rangle^2}{N(N+2)}$$

holds. $|D_N\rangle : \frac{1}{4}$ $\frac{N}{4}$ $\frac{N(N+2)}{16}$

Similar criterion for EPR steering.

# Violation of the criterion: entanglement is detected II




LHS/RHS for similar, but somewhat more complicated inequalities.
(top) Vitagliano *et al.*, Quantum 2024,
(bottom) Lange et *et al.*, Science 2018.

# Outline

# Particle number resolving detection

- The resolution of the particle number detection is not 1 particle. Typically, $\sim 10$.
- So far we did not need single particle resolution.
- Particle-number resolving detection could improve the detected quality of the state dramatically.
- We could also have new entanglement criteria relying on single particle resolution.
- It is possible to reach a single-particle resolution:

  M. Quensen, M. Hetzel, L. Santos, A. Smerzi, G. Tóth, L. Pezzé, C. Klempt, Hong-Ou-Mandel interference of more than 10 indistinguishable atoms, arXiv:2504.02691.

# Parity measurement

- We can measure the parititiy as

$$\langle \sigma_z \otimes \sigma_z \otimes .... \otimes \sigma_z \rangle = \langle f(J_z) \rangle ,$$

where

$$f(z) = e^{i2\pi(z+N/2)}.$$

- E. g, for $N = 4$, we have

$$\{f(z)\}_{z=-2,-1,0,1,2} = \{+1, -1, +1, -1, +1\}.$$

- Thus, we do not need individual access to the particles, but we need a particle number resolving detection.

# Entanglement conditions with many-body correlations

For separable states

$$\left|\langle\sigma_x\otimes\sigma_x\otimes...\otimes\sigma_x\rangle\right|+\left|\langle\sigma_y\otimes\sigma_y\otimes...\otimes\sigma_y\rangle\right|+\left|\langle\sigma_z\otimes\sigma_z\otimes...\otimes\sigma_z\rangle\right|\leq 1$$

holds.

- For the ideal Dicke state the value is 3.

| $N$ | $\langle\sigma_x^{\otimes N}\rangle$ | $\lvert\langle\sigma_z^{\otimes N}\rangle\rvert$ | $\langle J_x^2+J_y^2\rangle$ | $\mathcal{J}$ | $(\Delta J_z)^2$ |
|---|---|---|---|---|---|
| 2 | 0.892(22) | 0.965(13) | 1.892(22) | 0.946(11) | 0.0176(66) |
| 4 | 0.821(44) | 0.951(25) | 5.08(29) | 0.85(5) | 0.025(12) |
| 6 | 0.833(61) | 0.942(33) | 11.26(85) | 0.94(7) | 0.029(17) |
| 8 | 0.821(70) | 0.806(70) | 19.0(16) | 0.95(8) | 0.098(36) |
| 10 | 0.872(72) | 0.822(86) | 25.7(26) | 0.86(9) | 0.091(45) |
| 12 | 0.61(13) | 0.862(96) | 33.7(46) | 0.80(11) | 0.067(44) |

**Extended Data Table 1**: Measurement results for various particle numbers. The uncertainties denote one standard deviation.

# Proof

For separable states

$$\left|\langle\sigma_x \otimes \sigma_x \otimes ... \otimes \sigma_x\rangle\right| + \left|\langle\sigma_y \otimes \sigma_y \otimes ... \otimes \sigma_y\rangle\right| + \left|\langle\sigma_z \otimes \sigma_z \otimes ... \otimes \sigma_z\rangle\right| \leq 1$$

holds.

- *Proof.* For a product state of the type

  $$|\Psi^{(1)}\rangle \otimes |\Psi^{(2)}\rangle \otimes ... \otimes |\Psi^{(N)}\rangle$$

  the left-hand side can be bounded from above as

  $$\sum_{l=x,y,z}\left|\prod_{n=1}^{N}\left\langle\sigma_l^{(n)}\right\rangle\right| \leq \left|\left\langle\sigma_x^{(1)}\right\rangle\left\langle\sigma_x^{(2)}\right\rangle\right| + \left|\left\langle\sigma_y^{(1)}\right\rangle\left\langle\sigma_y^{(2)}\right\rangle\right| + \left|\left\langle\sigma_z^{(1)}\right\rangle\left\langle\sigma_z^{(2)}\right\rangle\right| \leq 1$$

  where in the first inequality we used that $\left|\left\langle\sigma_l^{(n)}\right\rangle\right| \leq 1$, and in the second inequality we used the Cauchy-Schwarz inequality and the fact that the length of the Bloch vector is at most one for a qubit.
- Separable states are mixtures of product states, hence the inequality is also valid for separable states. $\square$

# States detected

- The witness also detects the GHZ states as entangled.
- It also detects the singlet state given as

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \otimes ... \otimes \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

  has

$$(\Delta J_z)^2 = 0,$$

  and

$$\left\langle \sigma_x^{\otimes N} \right\rangle = 1, \quad \left\langle \sigma_y^{\otimes N} \right\rangle = 1,$$

  if $N$ is divisible by 4. This is a 2-entangled state.

- Thus, these operators cannot be used to detect genuine multipartite entanglement.

# Inequality with multi-particle correlations

**Observation 1.** For $N$-qubit quantum states,

$$\langle J_x\rangle^2/j^2 + \left\langle J_y\right\rangle^2/j^2 + \left\langle \sigma_z^{\otimes N}\right\rangle^2 \le 1$$

holds, where $j = N/2$ and

$$J_l = \frac{1}{2}\sum_{n=1}^{N} \sigma_l^{(n)}$$

for $l = x, y, z$.

*Proof.* The ground state of the Hamiltonian

$$H = BJ_x + K\sigma_z^{\otimes N},$$

where $B$ and $K$ are constants, is of the form

$$|\Psi\rangle = \alpha|0\rangle_x^{\otimes N} + \beta|1\rangle_x^{\otimes N},$$

which is a generalized GHZ state in the $x$-basis.

# Inequality with multi-particle correlations II

Then, the relevant expectation value of $J_x$ is

$$\langle J_x \rangle = \frac{N}{2} \langle \sigma_x \rangle_\phi$$

and the expectation value of the products of $\sigma_z$ matrices is

$$\left\langle \sigma_z^{\otimes N} \right\rangle = \langle \sigma_z \rangle_\phi \,,$$

where we define the single-qubit state

$$|\phi\rangle = \alpha|0\rangle_x + \beta|1\rangle_x.$$

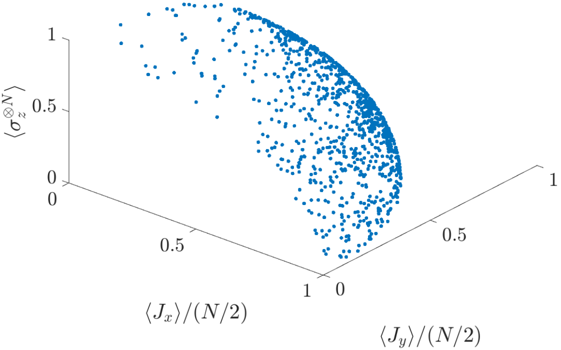
Since $\langle \sigma_x \rangle_\phi^2 + \langle \sigma_z \rangle_\phi^2 \leq 1$, it follows that

$$\langle J_x \rangle^2 / j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1.$$

Then, assuming that the mean spin is not parallel with the $x$-axis, but it is in the $xy$-plane, we arrive at our inequality. □

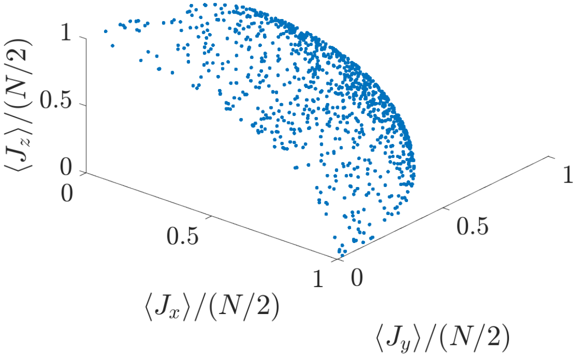
# Inequality with multi-particle correlations III

Generalized GHZ states:



# Inequality with multi-particle correlations IV

Comparison: spin coherent states



# Bipartite conditions

**Observation 2.** For bipartite separable states,

$$\langle J_x \otimes J_x \rangle / (j_1 j_2) + \left\langle J_y \otimes J_y \right\rangle / (j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N_1} \otimes \sigma_z^{\otimes N_2} \right\rangle \right| \leq 1$$

holds, where for the left half we have

$$j_1 = N_1/2, \qquad j_2 = N_2/2.$$

| | |
|---|---|
| $N_1$ particles | $N_2$ particles |

*Proof.* We start from Observation 1

$$\langle J_x \rangle^2 / j^2 + \left\langle J_y \right\rangle^2 / j^2 + \left\langle \sigma_z^{\otimes N} \right\rangle^2 \leq 1$$

and use the Cauchy-Schwarz inequality. □

# Bipartite conditions

- Problem: we need to measure observables in the two halves of the system.
- In many experiments, we measure only collective observables.
- We need to modify the inequality such that it works for that case.
- Note that we need to measure the particle number with a single particle resolution.

# Bipartite conditions

**Observation 3.** The following expression is true for bipartite separable states

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \le j(j+1)/(2j_1 j_2),$$

where

$$j_1 = N_1/2, \qquad j_2 = N_2/2, \qquad j = N/2.$$

*Proof.* We start from the previous Observation. We add to both sides

$$\left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle / (2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 \right\rangle / (2j_1 j_2).$$

Then follows the relation

$$\begin{aligned} &\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \\ &\le 1 + \left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle / (2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 \right\rangle / (2j_1 j_2). \end{aligned}$$

# Bipartite conditions II

Then, starting from the relation

$$\left\langle J_x^2 + J_y^2 \right\rangle /(2j_1 j_2) + \left|\left\langle \sigma_z^{\otimes N} \right\rangle\right| \\ \leq 1 + \left\langle (J_x^{(1)})^2 + (J_y^{(1)})^2 \right\rangle /(2j_1 j_2) + \left\langle (J_x^{(2)})^2 + (J_y^{(2)})^2 \right\rangle /(2j_1 j_2),$$

we use the inequality

$$\left\langle (J_x^{(n)})^2 + (J_y^{(n)})^2 \right\rangle \leq j_n(j_n + 1).$$

We arrive at

$$\left\langle J_x^2 + J_y^2 \right\rangle /(2j_1 j_2) + \left|\left\langle \sigma_z^{\otimes N} \right\rangle\right| \leq j(j+1)/(2j_1 j_2).$$

We need to measure only collective quantities! $\square$

# Outline

# Conditions for multi-particle entanglement

**Observation 4.** States violating the inequality

$$\left\langle J_x^2 + J_y^2 \right\rangle / (2j_1 j_2) + \left| \left\langle \sigma_z^{\otimes N} \right\rangle \right| \leq j(j+1)/(2j_1 j_2),$$

for

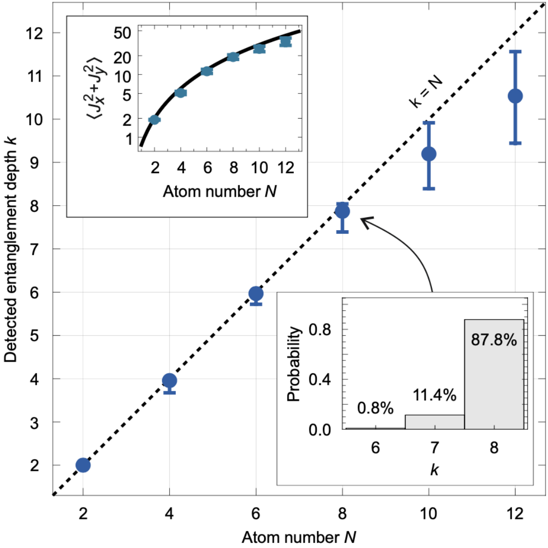
$$j_1 = k/2, \qquad j_2 = (N-k)/2$$

| | |
|---|---|
| $k$ particles | $N-k$ particles |

possess at least $(k+1)$-particle entanglement, where we assume that $k \geq N/2$.

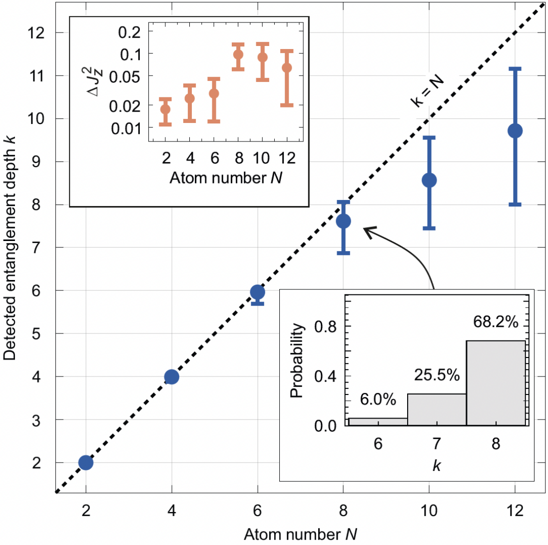
Violation for $k = N-1$ means genuine multipartite entanglement.

# Results



Detected entanglement depth $k$
Atom number $N$
$k = N$
$\langle J_x^2 + J_y^2 \rangle$
Atom number $N$
Probability
$k$
0.8%
11.4%
87.8%

# Comparison to the alternative method



# Conclusions

- We discussed how to detect bipartite and multipartite entanglement with many-body correlation measurements.
- The method has been successfully used in experiments with Dicke states up to 12 particles.
- It demonstrates the good quality of the created Dicke state.
- For the transparencies, see

www.gtoth.eu

- See also

M. Quensen, M. Hetzel, L. Santos,A. Smerzi,
G. Tóth, L. Pezzé, C. Klempt.

Hong-Ou-Mandel interference of more than 10 indistinguishable atoms,

arXiv:2504.02691.

THANK YOU FOR YOUR ATTENTION!