Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles

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Jan Mayen (NORWAY)
Greenland Sea
Denmark Strait
Barents Sea
Hammerfest
Murmansk
Tromsø
Kiruna
White Sea
Arkhangelsk
Norwegian Sea
Arctic Circle
Reykjavík
ICELAND
NORWAY
Luleå
Oulu
FINLAND
Umeå
SWEDEN
Trondheim
Gulf of Bothnia
Tampere
Lake Ladoga
Lake Onega
Tórshavn
Faroe Islands (DENMARK)
Helsinki
Turku
St. Petersburg
SHETLAND ISLANDS
Bergen
Oslo
Stockholm
ALAND ISLANDS
Gulf of Finland
Tallinn
ESTONIA
RUSSIA
ORKNEY ISLANDS
Rockall (U.K.)
HEBRIDES
Stavanger
Göteborg
Gotland
Riga
LATVIA
North Atlantic Ocean
Aberdeen
Glasgow
Edinburgh
North Sea
DENMARK
Copenhagen
Malmö
Baltic Sea
Öland
LITHUANIA
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Vitsyebsk
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Belfast
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Kaliningrad
RUSSIA
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Mahilyow
BELARUS
Dublin
IRELAND
Irish Sea
Isle of Man (U.K.)
Leeds
Liverpool
Manchester
Bornholm
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Homyel
Birmingham
Bremen
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Turin
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Belgrade
Varna
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Genoa
Sarajevo
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SAN MARINO
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Andorra la Vella
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Lisbon
SPAIN
Valencia
Balearic Sea
Naples
Sardinia
Tyrrhenian Sea
GREECE
Aegean Sea
BALEARIC ISLANDS
Athens
Sevilla
Mediterranean Sea
Cagliari
Ionian Sea
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Alborán Sea
Ceuta (SPAIN)
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles



Outline

1. **Motivation**
   - Why entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Detecting multipartite entanglement of Dicke states**
   - Dicke state realized with a BEC of two-state atoms
4. **Detecting bipartite entanglement of Dicke states**
   - Bipartite entanglement from multipartite entanglement in BEC
   - Dicke state in a double well
   - Problems to solve in an experiment
   - Entanglement detection in Dicke states

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea ...

Why is this challenging?

- It could happen that
  - it is not possible to create large scale entanglement in a system that is not completely isolated.
  - such entanglement is created, but we cannot verify its presence, since we can measure few things.

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Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

k-producibility/k-entanglement

A pure state is k-producible if it can be written as

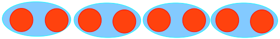
$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

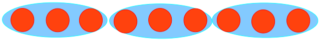
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.

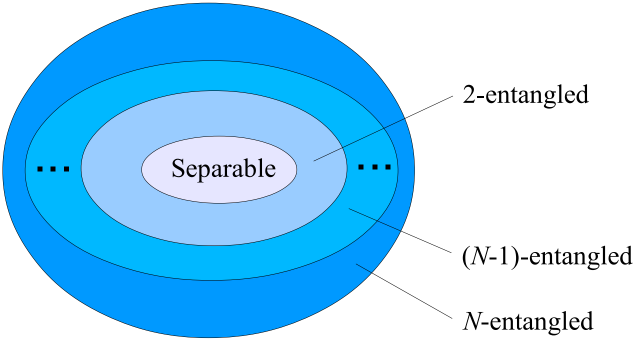
e.g., O. Gühne and GT, New J. Phys 2005.

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.





k-producibility/k-entanglement II



$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$ 2-entangled

$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$ 3-entangled

$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle)$ 4-entangled

k-entanglement means real k-particle quantumness

- k-entanglement means that we could not make trivially the experiment from $(k-1)$-particle experiments.
- The state is not a mixture of product states

$$\varrho_1 \otimes \varrho_2 \otimes \varrho_3 \otimes ...$$

such that all ϱ_l has at most $(k-1)$ qubits.

Outline

Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

Outline

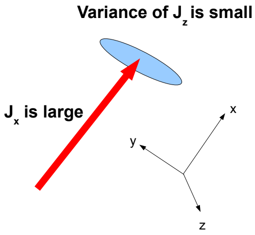
The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{l=1}^{M} |\psi_l\rangle,$$

where $|\psi_l\rangle$ is the state of at most k qubits.

Extreme spin squeezing

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

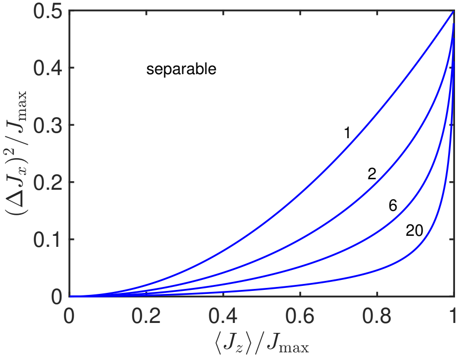
where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

Multipartite entanglement in spin squeezing II

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

Outline

Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_\mathrm{x}\rangle, \langle J_\mathrm{y}\rangle, \langle J_\mathrm{z}\rangle),$$
$$\vec{K} := (\langle J_\mathrm{x}^2\rangle, \langle J_\mathrm{y}^2\rangle, \langle J_\mathrm{z}^2\rangle).$$

- Then any state violating the following inequalities is entangled:

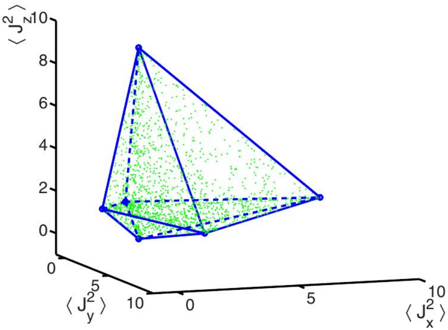
$$\langle J_x^2\rangle + \langle J_y^2\rangle + \langle J_z^2\rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \qquad \text{(singlet states)}$$
$$\langle J_k^2\rangle + \langle J_l^2\rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \qquad \text{(Dicke states)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2\rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

singlets: GT, Phys. Rev. A 69, 052327 (2004);
all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities depend only on two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Average 2-particle density matrix

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- We can detect states with a separable $\varrho_{2\mathrm{p}}$!

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Generalized spin squeezing criteria for $j = \frac{1}{2}$

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$$\vec{J} := (\langle J_{\rm x}\rangle, \langle J_{\rm y}\rangle, \langle J_{\rm z}\rangle),$$
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GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.

- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

  Summing over all permutations.
  Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;
GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.

- ... have high levels of bipartite entanglement

$$E_F \approx \log_2(N)/2 \equiv \log_2(\sqrt{N}).$$

Note that for a maximally entangled state,

$$E_F = \log_2(d).$$

J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi,
Phys. Rev. A 67, 022112 (2003).

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality. For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

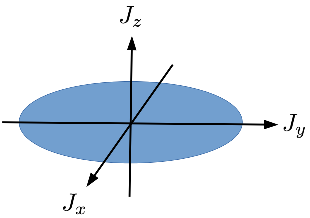
  holds.

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \mathrm{max.},$$

$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.
- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, New. J. Phys. (2014).

Multipartite entanglement - Our condition

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

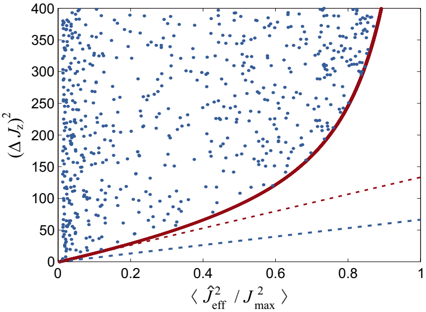
which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

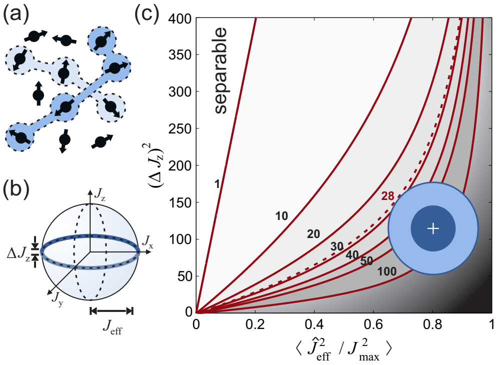
Concrete example



- $N = 8000$ particles, and $J_{\text{eff}} = J_x^2 + J_y^2$.
- Red curve: boundary for 28-particle entanglement.
- Blue dashed line: linear condition given in L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).
- Red dashed line: tangent of our curve.

Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J^2_{eff} = J^2_x + J^2_y, \quad J_{max} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014).

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$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Average 2-particle density matrix

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- We can even detect multipartite entanglement knowing only two-body correlations!

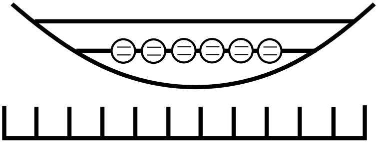
Outline

Bipartite entanglement from bosonic multipartite entanglement

- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

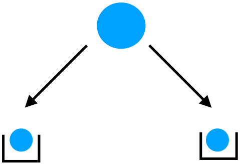
Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument



See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)

$|n_0 = 1\rangle|n_1 = 1\rangle$

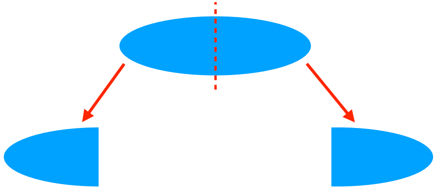


$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.

- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

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Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.

- Initially all atoms in the spin state $|j_z = 0\rangle$.

- Dynamics
$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$
Tunneling from mode 0 to the mode $+1$ and -1.

- Understanding the tunneling process
$$|j_z = 0\rangle|j_z = 0\rangle \to \frac{1}{\sqrt{2}}(|j_z = +1\rangle|j_z = -1\rangle + |j_z = -1\rangle|j_z = +1\rangle)$$
$=$ Dicke state of 2 particles.

Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N-2n, n, n\rangle.$$

- That is, $N-2n$ particles remained in the $|j_z = 0\rangle$ state, while $2n$ particles form a symmetric Dicke state given as

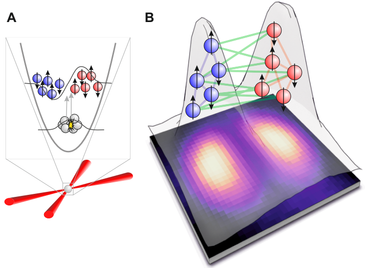
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}}\right),$$

  where we use $|0\rangle$ and $|1\rangle$ instead of $|j_z = -1\rangle$ and $|j_z = +1\rangle$.

- Half of the atoms in state $|0\rangle$, half of the atoms in state $|1\rangle$ + symmerization.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

Correlations for Dicke states

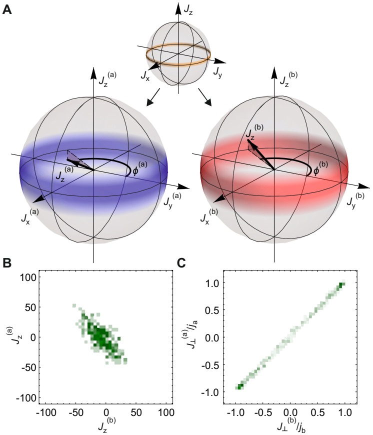
- For the Dicke state

$$(\Delta(J_x^a - J_x^b))^2 \approx 0,$$
$$(\Delta(J_y^a - J_y^b))^2 \approx 0,$$
$$(\Delta J_z)^2 = 0.$$

- Measurement results on well "*b*" can be predicted from measurements on "*a*"

$$\begin{aligned} J_x^b &\approx J_x^a, \\ J_y^b &\approx J_y^a, \\ J_z^b &= -J_z^a. \end{aligned}$$

Correlations for Dicke states - experimental results



Here, $J_{\perp}^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

Outline

1. Motivation
   - Why entanglement is important?
2. Spin squeezing and entanglement
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. Detecting multipartite entanglement of Dicke states
   - Dicke state realized with a BEC of two-state atoms
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   - Bipartite entanglement from multipartite entanglement in BEC
   - Dicke state in a double well
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Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Consider a density matrix

$$\varrho = \sum_{j_a, j_b} Q_{j_a, j_b} \varrho_{j_a, j_b},$$

  where ϱ_{j_a, j_b} are states with

$$N_a = 2j_a, \quad N_b = 2j_b$$

  particles in the two wells, Q_{j_a, j_b} are probabilities.
- ϱ is entangled iff at least one of the ϱ_{j_a, j_b} is entangled.

Problem 1: Varying particle number II

- Even if we have a constant total particle number, the ensemble will not be evenly split.

- Probability distribution for having $N/2 + x$ particles

$$p_x = 2^{-N}\binom{N}{N/2 + x}.$$

- Variance

$$\mathrm{var}(N_\mathrm{a}) = \mathrm{var}(x) = \langle x^2 \rangle = \frac{N}{4.}$$

- Collective variance

$$[\Delta(J_l^a - J_l^b)]^2 \approx \sum_{x=-N/2}^{N/2} p_x \left(\frac{N}{8} + \frac{1}{2}x^2\right) = \frac{N}{8} + \frac{1}{2}\mathrm{var}(x) = \frac{N}{4}, \quad l = x, y.$$

Twice as large due to the unequal splitting.

Problem 1: Varying particle number II

- $N/2 : N/2$ splitting:

$$[\Delta(J_l^a - J_l^b)]^2 = \frac{N}{8}$$

  for $l = x, y$.

- Real splitting with partition noise:

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}.$$

Problem 1: Varying particle number IV

- Let use the normalized quantity mentioned before

$$\mathcal{J}_l^- = \frac{1}{\sqrt{j_a(j_a+1)}} J_l^a - \frac{1}{\sqrt{j_b(j_b+1)}} J_l^b$$

  for $l = x, y$.

- For the variance of $\mathcal{J}_l$ we obtain for $N/2 + x : N/2 - x$ splitting

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{N}{N^2/2 + 4N - 2x^2}.$$

  After splitting $|x| \lesssim \sqrt{N/4}$.

- We have

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{2}{N}.$$

  $(\Delta \mathcal{J}_l^-)^2$ is not sensitive to the fluctuation of x if N is large.

Problem 2: States are not always symmertic in a BEC of two-state atoms

- Ideally, the BEC is in a single spatial mode.
- The state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.
- Our criterion must hadle this.

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Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x\rangle^2 + \left\langle J_y\right\rangle^2).$$

  Then,

$$(\Delta J_z)^2[(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4}[(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2\right\rangle + \left\langle J_y^2\right\rangle).$$

- Simple algebra yields

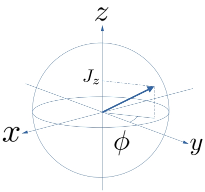
$$\left[(\Delta J_z)^2 + \frac{1}{4}\right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2\rangle + \langle J_y^2\rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2\rangle$ appears, not $\langle J_x\rangle^2$.

Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[(\Delta J_{\mathrm{z}})^2+\frac{1}{4}\right]}_{\sim\ \text{fluctuation of } J_z}\times\underbrace{\frac{(\Delta J_{\mathrm{x}})^2+(\Delta J_{\mathrm{y}})^2}{\langle J_{\mathrm{x}}^2\rangle+\langle J_{\mathrm{y}}^2\rangle}}_{\sim\ \text{phase fluctuation}}\geq\frac{1}{4}.$$



Handwaving description:
J_z and ϕ cannot be defined both with high accuracy.

Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_m^n = \frac{J_m^n}{\sqrt{j_n(j_n+1)}} \approx \frac{J_m^n}{N_n},$$

where $m = x, y$ and $n = a, b$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables → resistance to experimental imperfections.

Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_{\mathrm{z}})^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_{\mathrm{x}})^2 + (\Delta \mathcal{J}_{\mathrm{y}})^2\right] \geq \frac{1}{4}\left\langle \mathcal{J}_{\mathrm{x}}^2 + \mathcal{J}_{\mathrm{y}}^2 \right\rangle .$$

The two-well entanglement criterion

Suggestion of the experimentalists: we need a product criterion, since it is good for realistic noise.

Main result I

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{16}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle^2$$

holds.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_m^- &= \mathcal{J}_m^a - \mathcal{J}_m^b \end{aligned}$$

for $m = x, y$.

The two-well EPR-Steering criterion

Main result II

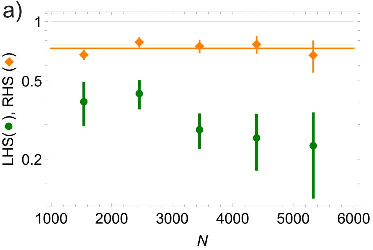
For states with a hidden state model,

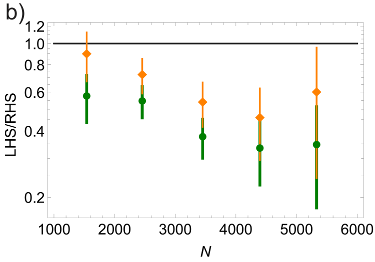
$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{4}\left\langle (\mathcal{J}_x^a)^2 + (\mathcal{J}_y^a)^2 \right\rangle^2$$

holds.

Any state violating the inequality cannot be described by a hidden state model, i.e., the state is *steerable*.

Violation of the criterion: entanglement is detected II





(a) LHS/RHS for Quantum 2023, and (b) for Science 2018.

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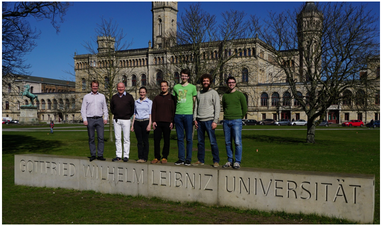
K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

Thanks to:



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Summary

- We discussed entanglement detection in particle ensembles.

THANK YOU FOR YOUR ATTENTION!

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