# Quantum states with a positive partial transpose are useful for metrology

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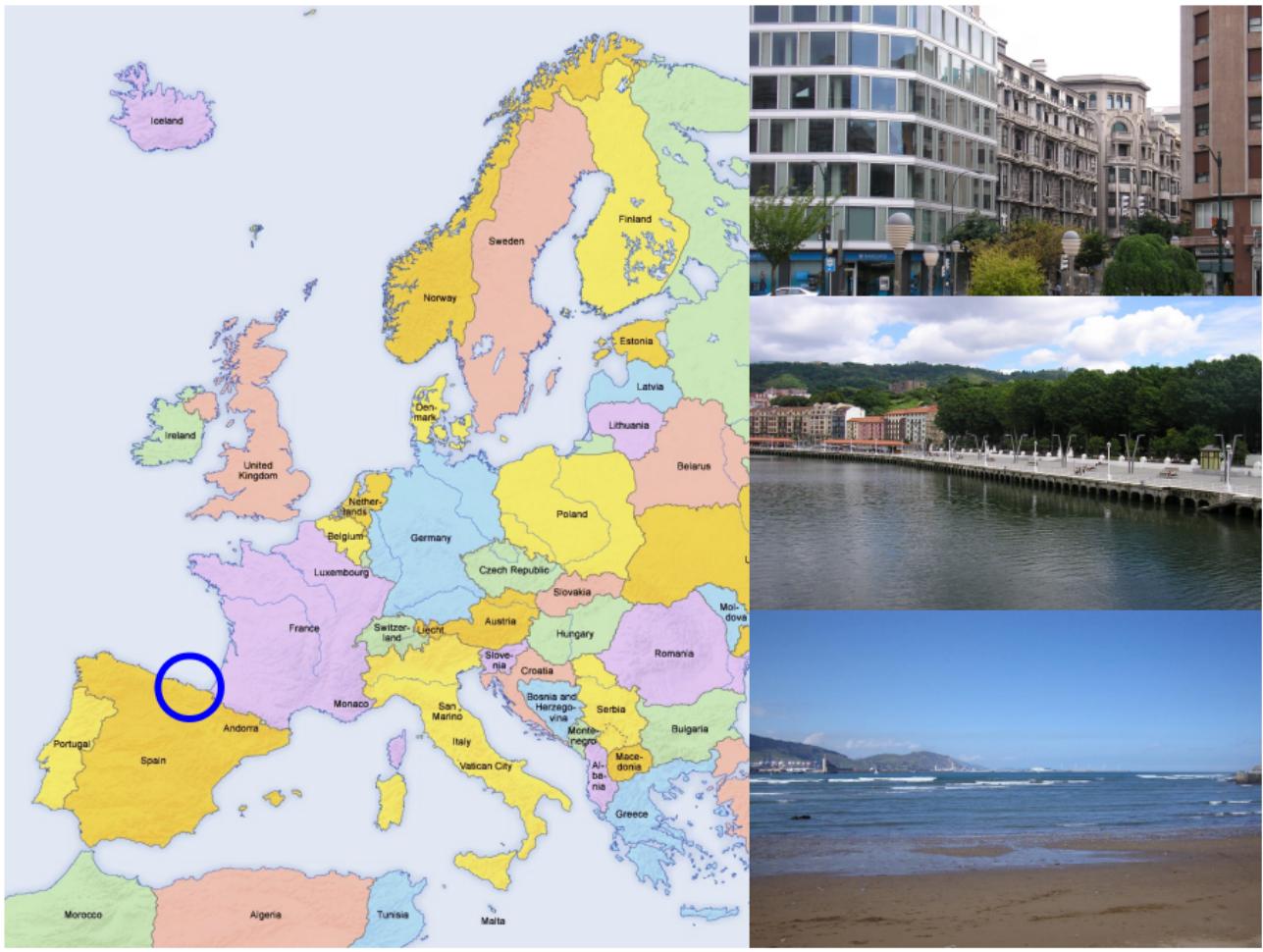
# Outline

## What are entangled states useful for?

- Entangled states are useful, but not all of them are useful for some task.

- Entanglement is needed for beating the shot-noise limit in quantum metrology.

- Intriguing question: Are states with a positive partial transpose useful for metrology? Can they also beat the shot-noise limit?

What are entangled states useful for?

PPT states
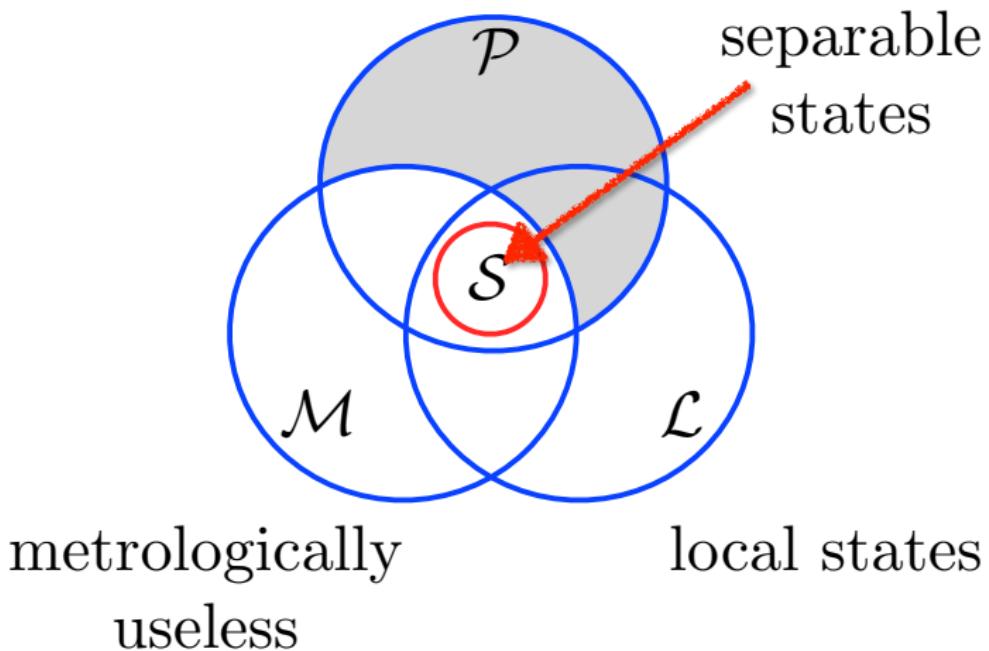
$\mathcal{P}$

separable states

$\mathcal{S}$

$\mathcal{M}$

$\mathcal{L}$

metrologically useless

local states

# Outline

# Quantum metrology

- Fundamental task in metrology

$$Q \quad \longrightarrow \quad \boxed{U(\theta) = \exp(-iA\theta)} \quad \longrightarrow \quad Q_\theta$$
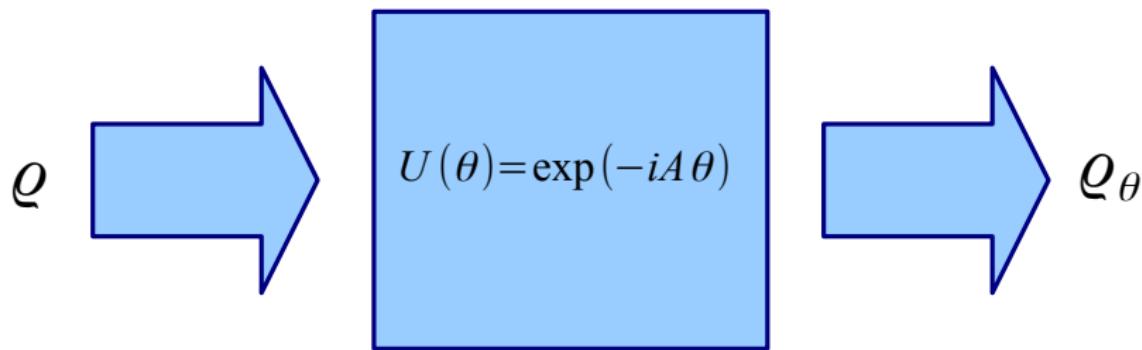
- We have to estimate $\theta$ in the dynamics

$$U = \exp(-iA\theta).$$

## Precision of parameter estimation

- Measure an operator $M$ to get the estimate $\theta$. The precision is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



$$\tan\alpha = \partial_\theta \langle M \rangle \big|_{\theta=0}$$
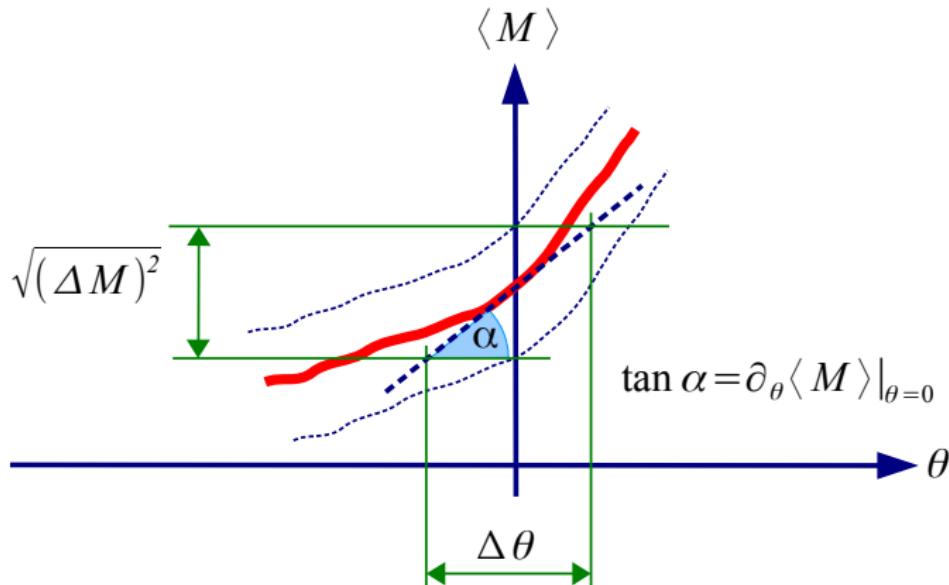
# The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{F_Q[\varrho, A]}, \qquad (\Delta\theta)^{-2} \leq F_Q[\varrho, A].$$

where $F_Q[\varrho, A]$ is the quantum Fisher information.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l \rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

# Special case $A = J_l$

- The operator $A$ is defined as

$$A = J_l = \sum_{n=1}^{N} j_l^{(n)}, \quad l \in \{x, y, z\}.$$

- Magnetometry with a linear interferometer
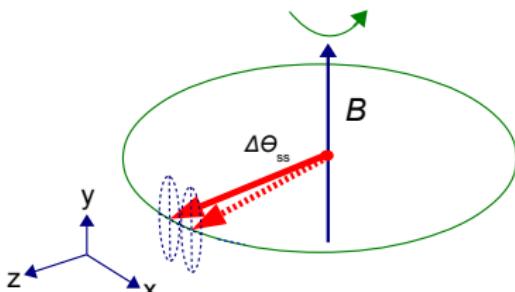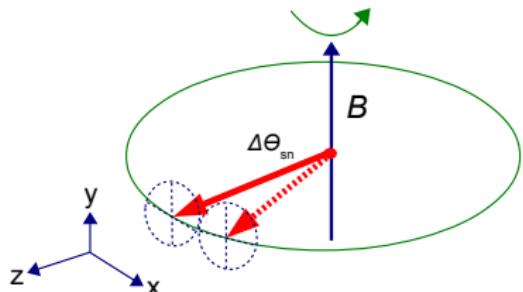
# Outline

# Properties of the Fisher information

Many bounds on the quantum Fisher information can be derived from these simple properties:

- For pure states, it equals four times the variance, $F[|\Psi\rangle\langle\Psi|, A] = 4(\Delta A)^2_\Psi$.

- For mixed states, it is convex.

# The quantum Fisher information vs. entanglement

- For separable states

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

[Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009); Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010)]

- For states with at most $k$-particle entanglement ($k$ is divisor of $N$)

$$F_Q[\varrho, J_l] \leq kN.$$

[P. Hyllus *et al.*, Phys. Rev. A 85, 022321 (2012); GT, Phys. Rev. A 85, 022322 (2012)].

- Macroscopic superpositions (e.g, GHZ states, Dicke states)

$$F_Q[\varrho, J_l] \propto N^2,$$

[F. Fröwis, W. Dür, New J. Phys. 14 093039 (2012).]

# Most important characteristics used for estimation

The quantum Fisher information is the convex roof of the variance

$$F_Q[\varrho, A] = 4 \min_{p_k, \Psi_k} \sum_k p_k (\Delta A)^2{}_k,$$

where

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

[GT, D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv1302.5311 (2013); GT, I. Apellaniz, J. Phys. A: Math. Theor. 47, 424006 (2014)]

- Thus, it is similar to entanglement measures that are also defined by convex roofs.

- Convex roof over purifications.
  [R. Demkowicz-Dobrzański, J. Kołodyński, M. Guţă, Nature Comm. 2012.

# Witnessing the quantum Fisher information based on few measurements

- Let us bound the quantum Fisher information based on some measurements.
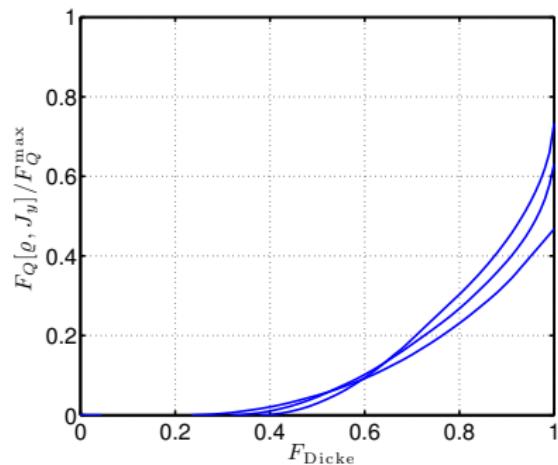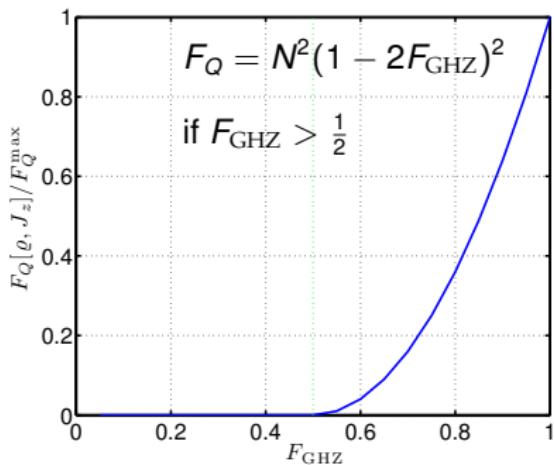


Quantum Fisher information vs. Fidelity with respect to (a) GHZ states and (b) Dicke states for $N = 4, 6, 12$.

[Apellaniz *et al.*, Phys. Rev. A 2017]

# Continuouity of QFI and QFI for symmetric states

- Arbitrarily small entanglement can be used to get close to Heisenberg scaling.
- The difference between the QFI of two states can be bounded by the distance of the two states.
- Bound on the QFI with the geometric measure of entanglement.

[R. Augusiak, J. Kołodyński, A. Streltsov, M. N. Bera, A. Acín, M. Lewenstein, PRA 2016]

- Random pure states of distinguishable particles typically do not lead to super-classical scaling of precision.
- Random states from the symmetric subspace typically achieve the optimal Heisenberg scaling.

[M. Oszmaniec, R. Augusiak, C. Gogolin, J. Kołodyński, A. Acín, M. Lewenstein, PRX 2016]

# Outline

# Results so far concerning metrologically useful PPT states

- Bound entangled states with PPT and some non-PPT partitions.
- Violates an entanglement criterion with three QFI terms.

  [ P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, and A. Smerzi, PRA 85, 022321 (2012). ]

- Non-unlockable bound entangled states with PPT and some non-PPT partitions.
- Violates the criterion with a single QFI term, better than shot-noise limit.

  [ Ł. Czekaj, A. Przysiężna, M. Horodecki, P. Horodecki, Phys. Rev. A 92, 062303 (2015). ]

on nonlocality [43]) to answer would be, Is there any family of quantum states that allows for a general Local Hidden Variables (LHV) model but can be used to obtain sub-shot-noise (i.e., better than classical) quantum metrology? This question is related to another question (especially in the context of both general requirements in quantum metrology [26] and recent results on nonlocality [43]) regarding whether there is any chance for sub-shot-noise metrology for states obeying the PPT condition with respect to *any* cut. While the present result

# Outline

## Our results

We look for bipartite PPT entangled states and multipartite states that are PPT with respect to all partitions.

## Maximizing the QFI for PPT states: brute force

- Maximize the QFI for PPT states. Remember

$$F_Q[\varrho, A] = 2 \sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

  where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

- Difficult to maximize a convex function over a convex set. The maximum is taken on the boundary of the set.

- Not guaranteed to find the global maximum.

- Note: Finding the *minimum* is possible!

## Maximizing the QFI for PPT state: our method

- We mentioned that the QFI gives a bound on the precision of the parameter estimation

$$F_Q[\varrho, A] \geq \frac{1}{(\Delta\theta)^2} = \frac{|\partial_\theta\langle M\rangle|^2}{(\Delta M)^2} = \frac{\langle i[M, A]\rangle^2}{(\Delta M)^2} \quad \text{(dynamics is } U = e^{-iA\theta}\text{)}.$$

- The bound is sharp

$$F_Q[\varrho, A] = \max_M \frac{\langle i[M, A]\rangle_\varrho^2}{(\Delta M)^2}.$$

[ M. G. Paris, Int. J. Quantum Inform. 2009. Used, e.g., in F. Fröwis, R. Schmied, and N. Gisin, 2015; I. Appelaniz *et al.*, NJP 2015.]

The maximum for PPT states can be obtained as

$$\max_{\varrho \text{ is PPT}} F_Q[\varrho, A] = \max_{\varrho \text{ is PPT}} \max_M \frac{\langle i[M, A]\rangle_\varrho^2}{(\Delta M)^2}.$$

Random operator $\rightarrow$ $M$

maximize over PPT states $\varrho$ for a given $M$

$\varrho$

maximize over $M$ for a given PPT state $\varrho$
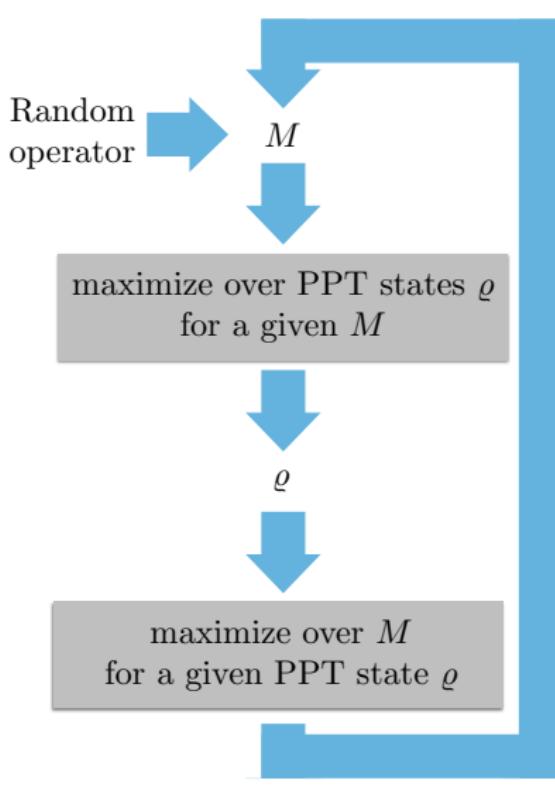
## Maximize over PPT states for a given M

Best precision for PPT states for a given operator $M$ can be obtained by a semidefinite program.

*Proof.*—Let us define first

$$f_M(X, Y) = \min_{\varrho} \quad \text{Tr}(M^2 \varrho),$$
$$\text{s.t.} \quad \varrho \geq 0, \varrho^{\text{T}k} \geq 0 \text{ for all } k, \text{Tr}(\varrho) = 1,$$
$$\langle i[M, A] \rangle = X \text{ and } \langle M \rangle = Y.$$

The best precsion for a given $M$ and for PPT states is

$$(\Delta \theta)^2 = \min_{X, Y} \frac{f_M(X, Y) - Y^2}{X^2}.$$

The state giving the best precision is $\varrho_{\text{PPTopt}}$.

Approach also maximzing over $\varrho$, for noisy states:
K. Macieszczak, arXiv:1312.1356v1 (thanks Janek for the reference!).

## Maximize over $M$ for a given PPT state

For a state $\varrho$, the best precision is obtained with the operator given by the symmetric logarithmic derivative

$$M = 2i \sum_{k,l} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|A|l\rangle,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

## Convergence of the method

The precision cannot get worse with the iteration!

# Convergence of the method II
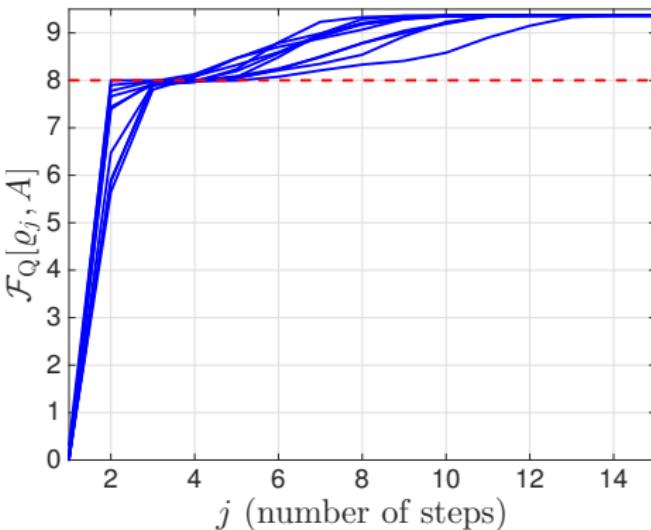


Generation of the $4 \times 4$ bound entangled state.

(blue)    10 attempts. After 15 steps, the algorithm converged.

(red)    Maximal quantum Fisher information for separable states.

# Robustness of the states

$$\varrho(p) = (1 - p)\varrho + p\varrho_{\mathrm{noise}}$$

- Robustness of entanglement: the maximal $p$ for which $\varrho(p)$ is entangled for any separable $\varrho_{\mathrm{noise}}$.
  [ Vidal and Tarrach, PRA 59, 141 (1999). ]

- Robustness of metrological usefulness: the maximal $p$ for which $\varrho(p)$ outperforms separable state for any separable $\varrho_{\mathrm{noise}}$.

| System | $A$ | $\mathcal{F}_Q[\varrho, A]$ | $\mathcal{F}_Q^{(\text{sep})}$ | $p_{\text{white noise}}$ |
|---|---|---|---|---|
| four qubits | $J_z$ | 4.0088 | 4 | 0.0011 |
| three qubits | $j_z^{(1)} + j_z^{(2)}$ | 2.0021 | 2 | 0.0005 |
| $2 \times 4$ (three qubits, only $1 : 23$ is PPT) | $j_z^{(1)} + j_z^{(2)}$ | 2.0033 | 2 | 0.0008 |

Multiqubit states

## Robustness of the states III

| $d$ | $\mathcal{F}_Q[\varrho, A]$ | $p_{\text{white noise}}$ | $p_{\text{noise}}^{\text{LB}}$ |
|---|---|---|---|
| 3 | 8.0085 | 0.0006 | 0.0003 |
| 4 | 9.3726 | 0.0817 | 0.0382 |
| 5 | 9.3764 | 0.0960 | 0.0361 |
| 6 | 10.1436 | 0.1236 | 0.0560 |
| 7 | 10.1455 | 0.1377 | 0.0086 |
| 8 | 10.6667 | 0.1504 | 0.0670 |
| 9 | 10.6675 | 0.1631 | 0.0367 |
| 10 | 11.0557 | 0.1695 | 0.0747 |
| 11 | 11.0563 | 0.1807 | 0.0065 |
| 12 | 11.3616 | 0.1840 | 0.0808 |

- $d \times d$ systems.
- Maximum of the quantum Fisher information for separable states is 8.
- The operator $A$ is not the usual $J_z$.

# Robustness of the states IV: $4 \times 4$ bound entangled PPT state

Let us define the following six states

$|\Psi_1\rangle = (|0,1\rangle + |2,3\rangle)/\sqrt{2}$, $|\Psi_2\rangle = (|1,0\rangle + |3,2\rangle)/\sqrt{2}$,

$|\Psi_3\rangle = (|1,1\rangle + |2,2\rangle)/\sqrt{2}$, $|\Psi_4\rangle = (|0,0\rangle + |3,3\rangle)/\sqrt{2}$,

$|\Psi_5\rangle = (1/2)(|0,3\rangle + |1,2\rangle) + |2,1\rangle/\sqrt{2}$,

$|\Psi_6\rangle = (1/2)(-|0,3\rangle + |1,2\rangle) + |3,0\rangle/\sqrt{2}$.

Our state is a mixture

$$\varrho_{4\times 4} = p\sum_{n=1}^{4} |\Psi_n\rangle\langle\Psi_n| + q\sum_{n=5}^{6} |\Psi_n\rangle\langle\Psi_n|,$$

where $q = (\sqrt{2} - 1)/2$ and $p = (1 - 2q)/4$. We consider the operator

$$A = H \otimes \mathbb{1} + \mathbb{1} \otimes H,$$

where $H = \mathrm{diag}(1, 1, -1, -1)$.

# Negativity

Apart from making calculations for PPT bound entangled states, we can also make calculations for states with given minimal eigenvalues of the partial transpose, or for a given negativity.

[ G. Vidal and R. F. Werner, PRA 65, 032314 (2002). ]

# Entanglement

| Bipartite state | Entanglement |
| --- | --- |
| $3 \times 3$ | 0.0003 |
| $4 \times 4$ | 0.0147 |
| $5 \times 5$ | 0.0239 |
| $6 \times 6$ | 0.0359 |
| $7 \times 7$ | 0.0785 |
| UPB $3 \times 3$ | 0.0652 |
| Breuer $4 \times 4$ | 0.1150 |

Convex roof of the linear entanglement entropy. The entanglement is also shown for the $3 \times 3$ state based on unextendible product bases (UPB) and for the Breuer state with a parameter $\lambda = 1/6$.

[ G. Tóth, T. Moroder, and O. Gühne, PRL 114, 160501 (2015). ]

# Metrologically useful quantum states with LHV models (PPT)

Consider the $2 \times 4$ state listed before. Possible to construct numerically a LHV model for the state.
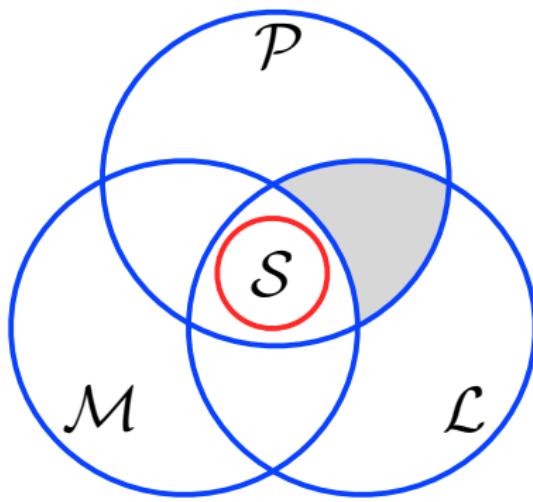


[ F. Hirsch, M. T. Quintino, T. Vértesi, M. F. Pusey, and N. Brunner, PRL 2016; D. Cavalcanti, L. Guerini, R. Rabelo, and P. Skrzypczyk, PRL 2016. ]

# Metrologically useful quantum states with LHV models (non-PPT)

- Two-qubit Werner state $p|\Psi^-\rangle\langle\Psi^-| + (1-p)\mathbb{1}/4$, with $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.
- Better for metrology than separable states ($\mathcal{F}_Q > 2$) for $p > 1 - 0.3596 = 0.6404$.
- They do not violate a Bell inequality for $p < 0.6829$.
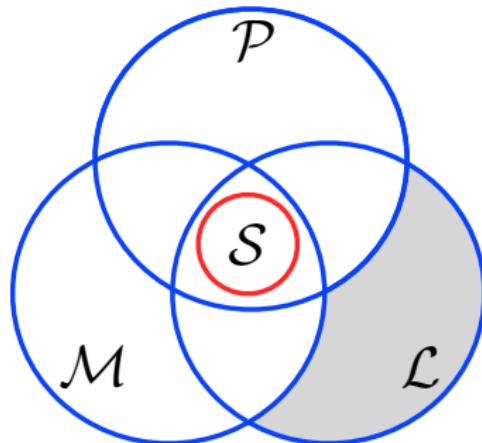


[ F. Hirsch, M. T. Quintino, T. Vértesi, M. Navascués, N. Brunner, Quantum 2017; A. Acín, N. Gisin, B. Toner, PRA 2006. ]

# Cluster states

Cluster states: resource in measurement-based quantum computing

[ R. Raussendorf and H. J. Briegel, PRL 2001. ]

- Fully entangled pure states.
- Violate a Bell inequality

  [ V. Scarani, A. Acín, E. Schenck, M. Aspelmeyer, PRA 2005; O. Gühne, GT, P. Hyllus, H. J. Briegel, PRL 2005; GT, O. Gühne, and H. J. Briegel, PRA 2006. ]

- Metrologically not useful

  [ P. Hyllus, O. Gühne, and A. Smerzi, PRA 2010. ]
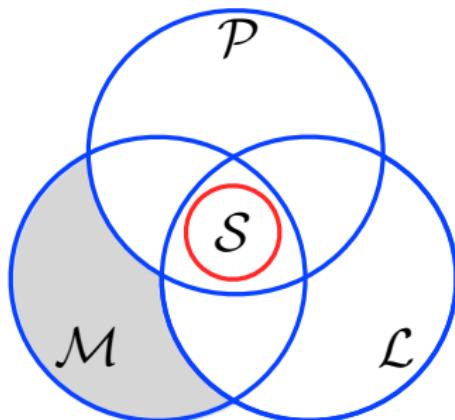
# Non-local PPT states

Counterexample for the Peres conjecture
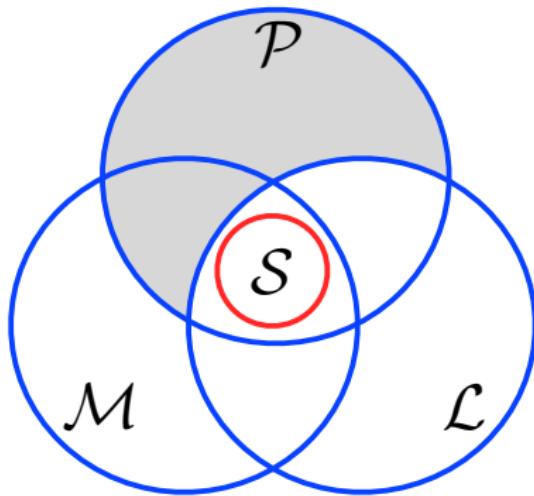


[ T. Vértesi and N. Brunner, Nature Communications 2015. ]

# Summary

- We presented quantum states with a positive partial transpose with respect to all bipartitions that are useful for metrology.

See:

Géza Tóth and Tamás Vértesi,

Quantum states with a positive partial transpose
are useful for metrology,

[arxiv:1709.03995](arxiv:1709.03995).

THANK YOU FOR YOUR ATTENTION!