

Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles

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Outline

1

Motivation

- Why entanglement is important?

2

Spin squeezing and entanglement

- Entanglement
- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for $j = \frac{1}{2}$

3

Entanglement detection close to singlet states

- Singlet obtained from squeezing of J_x, J_y , and J_z
- Applications for gradiometry

4

Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

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Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Dicke state in a double well
- Problems to solve in an experiment
- Entanglement detection in Dicke states

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea ...

Why is this challenging?

- It could happen that
 - it is not possible to create large scale entanglement in a system that is not completely isolated.
 - such entanglement is created, but we cannot verify its presence, since we can measure few things.

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Entanglement

A state is **(fully) separable** if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \dots \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is **entangled**.

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Many-particle systems for $j=1/2$

- For spin- $\frac{1}{2}$ particles, we can measure the **collective angular momentum operators**:

$$\mathbf{J}_l := \frac{1}{2} \sum_{k=1}^N \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the **expectation values** $\langle \mathbf{J}_l \rangle$.
- We can also measure the **variances**

$$(\Delta \mathbf{J}_l)^2 := \langle \mathbf{J}_l^2 \rangle - \langle \mathbf{J}_l \rangle^2.$$

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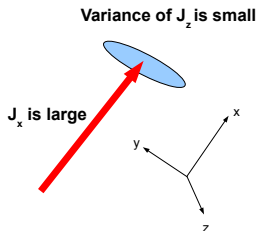
The standard spin-squeezing criterion

The **spin squeezing criterion for entanglement detection** is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

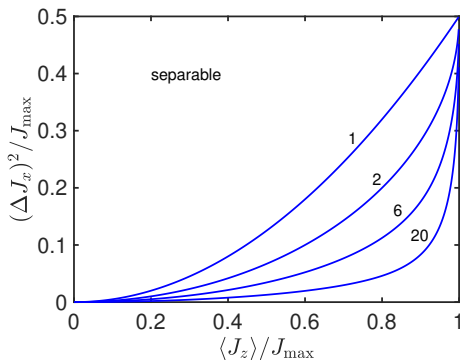
- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

Multipartite entanglement in spin squeezing

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test:
Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

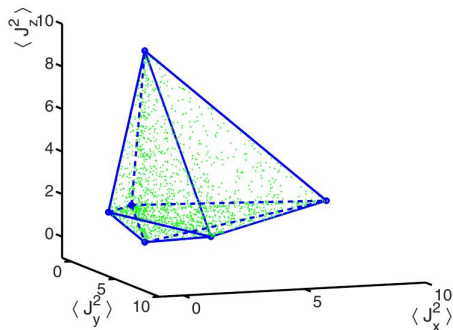
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \quad (\text{singlet states})$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \quad (\text{Dicke states})$$
$$(N-1) \left[(\Delta J_k)^2 + (\Delta J_l)^2 \right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z .

all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin- j : G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Generalized spin squeezing criteria for $j = \frac{1}{2} \mathbb{I}$

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

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Singlet state

- Singlet states are ground states of antiferromagnetic Hamiltonians.
- The permutationally invariant singlet is

$$\rho_{\text{singlet}} \propto \lim_{T \rightarrow 0} e^{-\frac{J_x^2 + J_y^2 + J_z^2}{T}}.$$

- Such a state has been created in cold atoms.

Singlet state experiments

- Creating singlets by squeezing the uncertainties of spin components → incoherent process, "creating entanglement with decoherence".



G. Tóth and M. W. Mitchell, Generation of macroscopic singlet states in atomic ensembles, *New J. Phys.* 12 053007 (2010).

- Entanglement in singlet states in $> 10^6$ cold Rb atoms.

N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, G.T., R. J. Sewell, and M. W. Mitchell, Generation of Macroscopic Singlet States in a Cold Atomic Ensemble, *Phys. Rev. Lett.* 113, 093601 (2014).

- Entanglement in singlet states in $> 10^{13}$ hot alkali atoms.

J. Kong, R. Jiménez-Martínez, C. Troullinou, V. G. Lucivero, G.T., and Morgan W. Mitchell, Measurement-induced, spatially-extended entanglement in a hot, strongly-interacting atomic system, *Nat. Commun.* 11, 2415 (2020).

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Applications of singlets

- Singlets are invariant under homogeneous fields.
- Gradient field destroys the singlet → Gradient metrology.

I. Apellaniz, I. Urizar-Lanz, Z. Zimborás, P. Hyllus, and GT, Precision bounds for gradient magnetometry with atomic ensembles, Phys. Rev. A 97, 053603 (2018).

S. Altenburg, M. Oszmaniec, S. Wölk, O. Gühne, Estimation of gradients in quantum metrology, Phys. Rev. A 96, 042319 (2017).

I. Urizar-Lanz, P. Hyllus, I. L. Egusquiza, M. W. Mitchell, and GT, Macroscopic singlet states for gradient magnetometry, Phys. Rev. A 88, 013626 (2013).

Detecting entanglement in the two-body density matrix

- Spin squeezing conditions with collective variables based on detecting entanglement in the two-body density matrix:

Spin Squeezing Inequalities and Entanglement of N Qubit States, J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. 95, 120502 (2005).

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Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z .
- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

Summing over all permutations. $\langle \vec{J}^2 \rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} (|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel *et al.*, PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al.*, Nat. Phys. 2012.;
L. Xin *et al.*, Phys. Rev. Lett 2023; Y-Q. Zhou, *et al.*, PNAS 2018.

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;

GT, PRA 2012;

GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.

- ... have high levels of bipartite entanglement

$$E_F \approx \log_2(N)/2 \equiv \log_2(\sqrt{N}).$$

Note that for a maximally entangled state,

$$E_F = \log_2(d).$$

J. K. Stockton, J. M. Geremia, A. C. Doherty, and H. Mabuchi,

Phys. Rev. A 67, 022112 (2003).

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality. For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

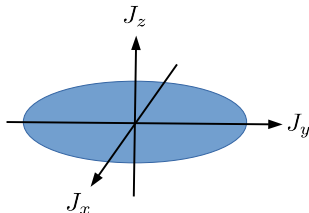
holds.

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2} \left(\frac{N}{2} + 1 \right) = \max.,$$

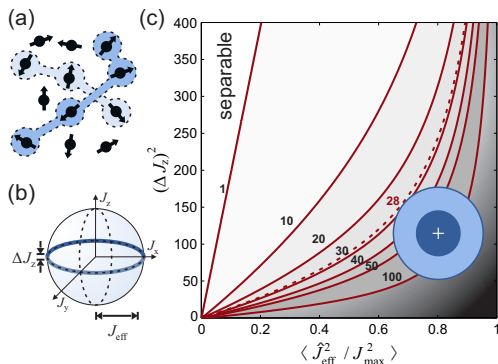
$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J_{\text{eff}}^2 = J_x^2 + J_y^2, \quad J_{\text{max}} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014); Vitagliano *et al.*, NJP 2017.

Multipartite entanglement

- Multi-particle entanglement in a Dicke state of hundreds of atoms:

Yi-Quan Zou *et al.*, PNAS 115 (25) 6381-6385 (2018),

Lin Xin *et al.*, Phys. Rev. Lett. 131, 133402 (2023).

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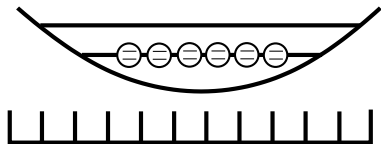
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Bipartite entanglement from bosonic multipartite entanglement

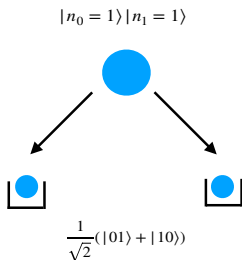
- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument

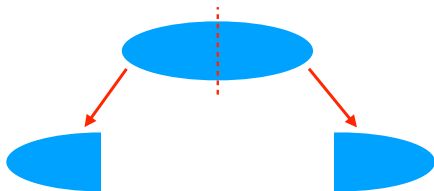


See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)



Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.
- **The splitting does not generate entanglement**, if we consider projecting to a fixed particle number.



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Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.
- Initially all atoms in the spin state $|j_z = 0\rangle$.

- Dynamics

$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$

Tunneling from mode 0 to the mode +1 and -1.

- Understanding the tunneling process

$$\begin{aligned} |j_z = 0\rangle |j_z = 0\rangle &\rightarrow \frac{1}{\sqrt{2}} (|j_z = +1\rangle |j_z = -1\rangle + |j_z = -1\rangle |j_z = +1\rangle) \\ &= \text{Dicke state of 2 particles.} \end{aligned}$$

Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |n_{\text{total}} - 2n, n, n\rangle.$$

- That is, $n_{\text{total}} - 2n$ particles remained in the $|j_z = 0\rangle$ state, while $2n$ particles form a symmetric Dicke state given as

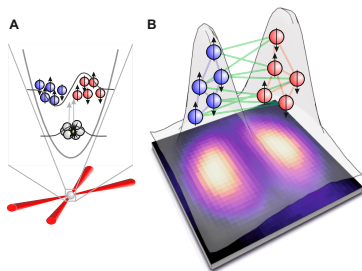
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

where we use $|0\rangle$ and $|1\rangle$ instead of $|j_z = -1\rangle$ and $|j_z = +1\rangle$, and $N = 2n$.

- Half of the atoms in state $|0\rangle$, half of the atoms in state $|1\rangle$ + symmerization.

Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, and C. Klempt, Entanglement between two spatially separated atomic modes, *Science* 360, 416 (2018).

Correlations for Dicke states

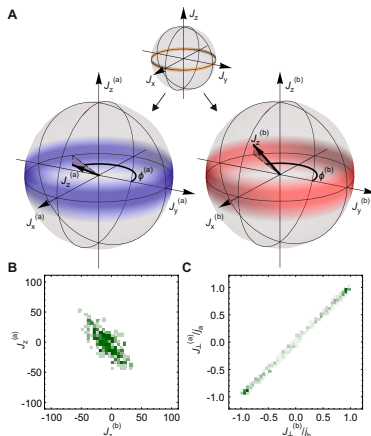
- For the Dicke state

$$\begin{aligned}(\Delta(J_x^a - J_x^b))^2 &\approx 0, \\(\Delta(J_y^a - J_y^b))^2 &\approx 0, \\(\Delta J_z)^2 &= 0.\end{aligned}$$

- Measurement results on well "b" can be predicted from measurements on "a"

$$\begin{aligned}J_x^b &\approx J_x^a, \\J_y^b &\approx J_y^a, \\J_z^b &= -J_z^a.\end{aligned}$$

Correlations for Dicke states - experimental results



$$\text{Here, } J_{\perp}^{(n)} = \cos \alpha J_X^{(n)} + \sin \alpha J_Y^{(n)}.$$

Experiment in [K. Lange *et al.*, Science 334, 773–776 \(2011\).](#)

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Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.
- Postselecting for a given particle number is not feasible.
- Consider a density matrix

$$\rho = \sum_{j_a, j_b} Q_{j_a, j_b} \rho_{j_a, j_b},$$

where ρ_{j_a, j_b} are states with

$$N_a = 2j_a, \quad N_b = 2j_b$$

particles in the two wells, Q_{j_a, j_b} are probabilities.

- ρ is entangled iff at least one of the ρ_{j_a, j_b} is entangled.

Problem 1: Varying particle number II

- Even if we have a constant total particle number, the ensemble will not be evenly split.
- Probability distribution for having $N/2 + x$ particles

$$p_x = 2^{-N} \binom{N}{N/2 + x}.$$

- Variance

$$\text{var}(N_a) = \text{var}(x) = \langle x^2 \rangle = \frac{N}{4}.$$

- Collective variance

$$[\Delta(J_l^a - J_l^b)]^2 \approx \sum_{x=-N/2}^{N/2} p_x \left(\frac{N}{8} + \frac{1}{2}x^2 \right) = \frac{N}{8} + \frac{1}{2}\text{var}(x) = \frac{N}{4}, \quad l = x, y.$$

Twice as large due to that the ensemble is not evenly split.

Problem 1: Varying particle number III

- $N/2 : N/2$ splitting:

$$[\Delta(J_l^a - J_l^b)]^2 = \frac{N}{8}$$

for $l = x, y$.

- Real splitting with partition noise:

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}.$$

Problem 1: Varying particle number IV

- Let us use the normalized quantity mentioned before

$$\mathcal{J}_l^- = \frac{1}{\sqrt{j_a(j_a + 1)}} J_l^a - \frac{1}{\sqrt{j_b(j_b + 1)}} J_l^b \approx 2 \left(\frac{J_l^a}{N_a} - \frac{J_l^b}{N_b} \right)$$

for $l = x, y$.

- For the variance of \mathcal{J}_l we obtain for $N/2 + x : N/2 - x$ splitting

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{N}{N^2/2 + 4N - 2x^2}.$$

After splitting $|x| \lesssim \sqrt{N/4}$.

- We have

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{2}{N}.$$

$(\Delta \mathcal{J}_l^-)^2$ is not sensitive to the fluctuation of x if N is large.

Problem 2: States are not always symmetric in a BEC of two-state atoms

- Ideally, the BEC is in a single spatial mode.
- The state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, so there is no perfect symmetry.
- Our criterion must handle this.

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- Collective measurements
- The original spin-squeezing criterion
- Generalized criteria for $j = \frac{1}{2}$

3

Entanglement detection close to singlet states

- Singlet obtained from squeezing of J_x, J_y , and J_z
- Applications for gradiometry

4

Detecting multipartite entanglement of Dicke states

- Dicke state realized with a BEC of two-state atoms

5

Detecting bipartite entanglement of Dicke states

- Bipartite entanglement from multipartite entanglement in BEC
- Dicke state in a double well
- Problems to solve in an experiment
- Entanglement detection in Dicke states

Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2 [(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4} (\langle J_x \rangle^2 + \langle J_y \rangle^2).$$

Then,

$$(\Delta J_z)^2 [(\Delta J_x)^2 + (\Delta J_y)^2] + \frac{1}{4} [(\Delta J_x)^2 + (\Delta J_y)^2] \geq \frac{1}{4} (\langle J_x^2 \rangle + \langle J_y^2 \rangle).$$

- Simple algebra yields

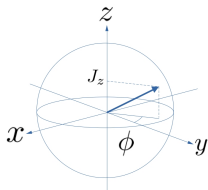
$$\left[(\Delta J_z)^2 + \frac{1}{4} \right] \times \frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2 \rangle$ appears, not $\langle J_x \rangle^2$.

Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[(\Delta J_z)^2 + \frac{1}{4} \right]}_{\sim \text{fluctuation of } J_z} \times \underbrace{\frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle}}_{\sim \text{phase fluctuation}} \geq \frac{1}{4}.$$



Handwaving description:

J_z and ϕ cannot be defined both with high accuracy.

Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_m^n = \frac{J_m^n}{\sqrt{j_n(j_n + 1)}} \approx \frac{J_m^n}{N_n},$$

where $m = x, y$ and $n = a, b$ (i.e., **left** well, **right** well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables \rightarrow resistance to experimental imperfections.

Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_z)^2 + \frac{1}{4} \right] \left[(\Delta \mathcal{J}_x)^2 + (\Delta \mathcal{J}_y)^2 \right] \geq \frac{1}{4} \langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \rangle.$$

The two-well entanglement criterion

Suggestion of the experimentalists: we need a product criterion, since it is good for realistic noise.

Main result I

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4} \right] \left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2 \right] \geq \frac{1}{16} \langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \rangle^2$$

holds.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_m^- &= \mathcal{J}_m^a - \mathcal{J}_m^b \end{aligned}$$

for $m = x, y$.

The two-well EPR-Steering criterion

Main result II

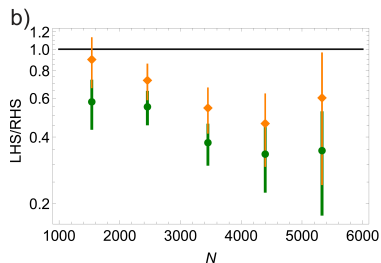
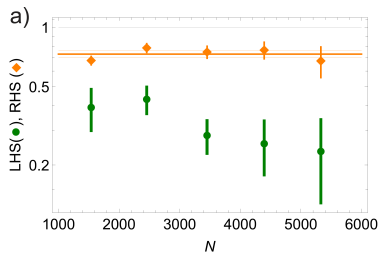
For states with a hidden state model,

$$\left[(\Delta J_z)^2 + \frac{1}{4} \right] \left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2 \right] \geq \frac{1}{4} \langle (\mathcal{J}_x^a)^2 + (\mathcal{J}_y^a)^2 \rangle^2$$

holds.

Any state violating the inequality cannot be described by a hidden state model, i.e., the state is *steerable*.

Violation of the criterion: entanglement is detected II



(a) LHS/RHS for Quantum 2023, and (b) for Science 2018.

G. Vitagliano, I. Apellaniz, M. Fadel, M. Kleinmann, B. Lücke, C. Klempt, GT, Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles, *Quantum* 7, 914 (2023).

K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, C. Klempt, Entanglement between two spatially separated atomic modes, *Science* 360, 416 (2018).

Recent related works published back to back

- Spatially distributed multipartite entanglement enables EPR steering of atomic clouds:

P. Kunkel, M. Prüfer, H. Strobel, D. Linnemann, A. Frölian, T. Gasenzer, M. Gärtner, M. K. Oberthaler, Science 360, 413-416 (2018).

- Spatial entanglement patterns and Einstein-Podolsky-Rosen steering in Bose-Einstein condensates:

M. Fadel, T. Zibold, B. Décamps, P. Treutlein, Science 360, 409-413 (2018).

Thanks to:

- Photonic experiments: N. Kiesel, W. Wieczorek, L. Knips, H. Weinfurter, MPQ München
- Singlet state experiments with cold atoms: R. J. Sewell, N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, J. Kong, R. Jiménez-Martínez, C. Troullinou, V. G. Lucivero, M. W. Mitchell, ICFO, Barcelona
- Dicke state experiments with cold atoms: I. Kruse, J. Peise, K. Lange, B. Lücke, C. Klempt, Hannover U., Germany; J. Arlt, Arhus U., Denmark
- Theory: O. Gühne, Siegen U., Germany
- C. Knapp, H.-J. Briegel, Innsbruck, Austria
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Summary

- We discussed entanglement detection in particle ensembles close to singlet states and Dicke states.
- We also discussed the detection of bipartite entanglement in Dicke states.

THANK YOU FOR YOUR ATTENTION!

See G. Vitagliano, M. Fadel, I. Apellaniz, M. Kleinmann, B. Lücke, C. Klempt, G. Tóth, *Quantum* 7, 914 (2023).

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