Detection of multipartite entanglement close to symmetric Dicke states

G. Tóth[1,2,3]

Collaboration:

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Denmark Strait
Barents Sea
Hammerfest
Murmansk
Tromsø
Kiruna
White Sea
Arkhangel
Norwegian Sea
Arctic Circle
Reykjavík
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Lake Onega
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Riga
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Sevilla
Mediterranean Sea
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Ionian Sea
Palermo
Gibraltar (U.K.)
Málaga
Sicily
Strait of Gibraltar
Alborán Sea
Ceuta (SPAIN)
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Rabat
Valletta
MALTA
TUNISIA
Casablanca
ALGERIA
MOROCCO
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
300 Kilometers
300 Miles

Outline

1. **Motivation**
   - Why multipartite entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. **Spin squeezing for Dicke states**
   - Entanglement detection close to Dicke states
   - Detection of multipartite entanglement close to Dicke states
   - Our conditions are stronger than the original conditions

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Only collective quantities can be measured.
- Full tomography is not possible, we still have to say something meaningful.
- Claiming “entanglement” is not sufficient for many particles.

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Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

k-producibility/k-entanglement

A pure state is *k-producible* if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle....$$

where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.
[e.g., O. Gühne and G. Tóth, New J. Phys 2005.]

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.

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Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

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The standard spin-squeezing criterion

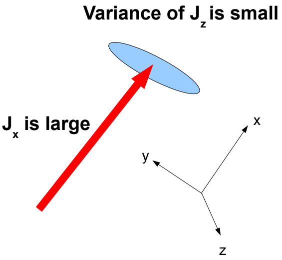
- The spin squeezing criteria for entanglement detection is

$$\xi_s^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

If $\xi_s^2 < 1$ then the state is entangled.

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- States detected are like this:



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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2},$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2},$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

[GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007)]

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Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality

$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}.$$

- It detects states close to symmetric Dicke states with $\langle J_z \rangle = 0$ defined as

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right),$$

since for these states we have

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \mathrm{max.},$$

$$\langle J_z^2 \rangle = 0.$$

Spin Squeezing Inequality for Dicke states II

- Dicke states of this type are very good for metrology. For the quantum Fisher information

$$F_Q[\varrho, J_x] + F_Q[\varrho, J_y] + F_Q[\varrho, J_z] = 4 \times \frac{N}{2}\left(\frac{N}{2} + 1\right) = \max.$$

- GHZ states also fulfill this condition!

- Note: For k-producible states

$$F_Q[\varrho, J_x] + F_Q[\varrho, J_y] + F_Q[\varrho, J_z] \lesssim (k+2)N,$$

$$F_Q[\varrho, J_l] \lesssim kN.$$

[Collaboration of P. Hyllus, L. Pezzé, and A. Smerzi: P. Hyllus et al., PRA 2012; B. Lücke et al, Science 2011.]

[G. Tóth, PRA 2012; see also G. Tóth, I. Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”.]

Spin Squeezing Inequality for Dicke states III

Based on the above inequality, let us define a new spin squeezing parameter

$$\xi_{\mathrm{os}}^2 = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2\rangle - \frac{N}{2}}.$$

[G. Vitagliano, I. Apellaniz, I.L. Egusquiza, and GT, PRA (2014)]

- For the symmetric Dicke state with $\langle J_z\rangle = 0$, the numerator is minimal, the denominator is maximal.

- The original spin squeezing parameter would not detect the Dicke state as entangled, since

$$\xi_{\mathrm{s}}^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2} = N\frac{(\Delta J_z)^2}{0}.$$

Fully polarized states

- Relation between the second moments and the expectation value

$$\langle J_x^2 \rangle = \langle J_x \rangle^2 + (\Delta J_x)^2 \geq \langle J_x \rangle^2.$$

- For states polarized in the x-direction and spin squeezed along the z-direction, for $N \gg 1$, we have

$$\langle J_x^2 \rangle \approx \langle J_x \rangle^2 \gg N.$$

Hence, for fully polarized states

$$\xi_{\rm os}^2 = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2 \rangle - \frac{N}{2}} \approx \xi_{\rm s}^2 = N\frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

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Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{n=1}^{M} |\psi^{(n)}\rangle,$$

where $|\psi^{(n)}\rangle$ is the state of at most k qubits.

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x\rangle^2 + \langle J_y\rangle^2}}{J_{\max}}\right),$$

where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x\rangle}{j}=X} (\Delta j_z)^2.$$

[A. S. Sørensen and K. Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: C. Gross, T. Zibold, E. Nicklas, J. Esteve, M. K. Oberthaler, Nature 464, 1165 (2010).]

Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.

Multipartite entanglement around Dicke states II

- Sørensen-Mølmer condition for *k*-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

which is true for pure *k*-producible states.

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

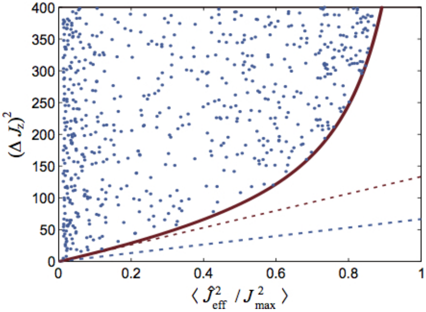
Multipartite entanglement around Dicke states III

- For large N, and $k \ll N$ we have

$$(\Delta J_z)^2 \gtrsim J_{\max} F_{\frac{k}{2}}\left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle}}{J_{\max}}\right).$$

Concrete example

- Let us draw the boundary of k-producible states.



- For $N = 8000$ particles, state below the curve have a larger than 28-particle entanglement.
- The blue dashed line is the condition given in [L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).]
- The red dashed line is the tangent of our curve.

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Our condition is stronger

- Examine, when our spin squeezing parameter is stronger:

$$\xi_{\rm os}^2 = (N-1)\frac{(\Delta J_z)^2}{\langle J_x^2 + J_y^2\rangle - \frac{N}{2}} < \xi_{\rm s}^2 = N\frac{(\Delta J_z)^2}{\langle J_x\rangle^2 + \langle J_y\rangle^2}.$$

- Noisy states of the form

$$\varrho_{\rm noisy} = (1-p)\varrho + p\frac{\mathbb{1}}{2^N}.$$

- For this state,

$$\left(\langle J_x^2 + J_y^2\rangle_{\rm noisy} - \tfrac{N}{2}\right) = (1-p)\left(\langle J_x^2 + J_y^2\rangle - \tfrac{N}{2}\right),$$
$$\left(\langle J_x\rangle^2 + \langle J_y\rangle^2\right)_{\rm noisy} = (1-p)^2\left(\langle J_x\rangle_\varrho^2 + \langle J_y\rangle_\varrho^2\right).$$

- Hence, $\xi_{\rm os}^2 < \xi_{\rm s}^2$ if

$$(\Delta J_x)^2 + (\Delta J_y)^2 > \tfrac{N}{2} - p\left(\langle J_x\rangle_\varrho^2 + \langle J_y\rangle_\varrho^2\right).$$

Thus, in all practical cases our relation is stronger for large N : fully polarized states with $\langle J_x\rangle_\varrho^2 + \langle J_y\rangle_\varrho^2 > O(N)$ and Dicke states.

Our condition is stronger II

- We can also incorporate the original spin squeezing parameter using

$$\left(\langle J_x\rangle^2 + \langle J_y\rangle^2\right) = \frac{1}{\xi_s^2} N(\Delta J_z)^2. \quad (1)$$

- Hence, $\xi_{os}^2 < \xi_s^2$ if

$$(\Delta J_x)^2 + (\Delta J_y)^2 > N\left(\frac{1}{2} - p\frac{(\Delta J_z)^2}{\xi_s^2}\right).$$

- Assuming $\xi_s < 1$, the right-hand side is negative for $p > 0$ unless we have $(\Delta J_z)^2 \sim O(N^0)$. Not realistic.

Hence, for large N, if $\xi_s < 1$ then (to a very good degree of approximation)

$$\xi_{os}^2 \leq \xi_s^2.$$

[G. Vitagliano, I. Apellaniz, I.L. Egusquiza, and GT, PRA (2014)]

Our condition is stronger - multipartite case

- Our entanglement condition is stronger if

$$\langle J_x^2 + J_y^2 \rangle - J_{\max}(\tfrac{k}{2} + 1) \geq \langle J_x \rangle^2 + \langle J_y \rangle^2.$$

- Noisy states of the form

$$\varrho_{\text{noisy}} = (1-p)\varrho + p\frac{\mathbb{1}}{2^N}.$$

- Our entanglement condition is stronger if

$$(\Delta J_x)^2 + (\Delta J_y)^2 \geq \tfrac{N}{2}(\tfrac{k}{2} + 1) - p\Big(\langle J_x \rangle_\varrho^2 + \langle J_y \rangle_\varrho^2\Big).$$

- Thus, in all practical cases our relation is stronger for large N :
  - fully polarized states with $\langle J_x \rangle_\varrho^2 + \langle J_y \rangle_\varrho^2 \sim O(N^q)$ with $q > 1$,
  - Dicke states with $(\Delta J_x)^2 + (\Delta J_y)^2 \sim O(N^2)$.
- Similar argument, as before for $\xi_s < 1$.

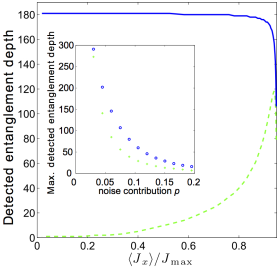
Our condition is stronger - multipartite case II

- Consider spin squeezed states as ground states of

$$H(\Lambda) = J_z^2 - \Lambda J_x.$$

For $\Lambda = \infty$, the ground state is fully polarized. For $\Lambda = 0$, it is the symmetric Dicke state.

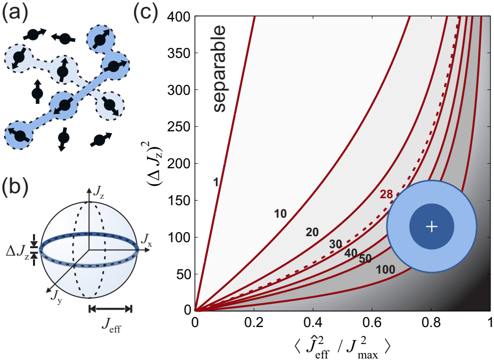
- Our condition vs. original condition for N=4000 and p=0.05



[B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, Phys. Rev. Lett. 112, 155304 (2014).]

Experimental results

- Bose-Einstein condenstate, 8000 particles. 28-particle entanglement is detected.



[B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, Phys. Rev. Lett. 112, 155304 (2014).]

Summary

- We showed how to detect multipartite entanglement close to Dicke states, by measuring collective quantities only.
- The condition is optimal: it detects all entangled states that can be detected based on the measured quantities.

G. Vitagliano, I Apellaniz, I.L. Egusquiza, and G. Tóth, PRA (2014).

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos,
G. Tóth, and C. Klempt, Phys. Rev. Lett. 112, 155304 (2014).
Editors' Suggestion, synopsis at physics.aps.org, featured in the
Revista Española de Física.

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